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Abstract

Barro's model is an AK model, and there cannot be dynamic inefficiency since the social yield of the capital is higher than the growth rate. But it may be that the private yield and thus the interest rate are lower than the growth rate. One can thus have a Ponzi game and the government can allow a permanent roll-over of debt and cut taxes. However we show that in this model since the capital is under-accumulated, playing a Ponzi game produces a crowding-out of capital and reduces the growth rate and welfare. The practical message of this article is that even when the interest rate is lower than the growth rate, the public debt is not a Pareto improvement when it generates a crowding-out of capital and reduces endogenous growth.

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1. Introduction

A Ponzi game allows the government to finance its spending through debt, not taxes. The tax cut, other things being equal, increases the welfare. If public expenditures are productive, they can increase the growth rate and allow the realization of a Ponzi game. We then have the situation where we "pay today's public spending with tomorrow's economic growth." We examine this situation. Literature on the subject leads us to clarify two issues.

McCallum (1984) showed that in the representative agent model, the government can never play a Ponzi game (i.e. allow debt to grow at a rate above the interest rate) because the representative agent never want to grow their loan at that rate. This is so because the agent must satisfy the transversality condition. This means not accumulating capital at a rate greater than r so that in an infinite time the present value of its capital is zero. Tirole (1985), O'Connell and Zeldes (1988) showed that in the overlapping generations (OLG) model, even if each agent satisfies its transversality condition (in this context, die after consuming all its capital) it does not prevent the government from playing a Ponzi game. Even if each individual satisfies its transversality condition, growth rate higher than the interest rate (possible in this model), will mean that overall savings will grow at a rate higher than r. The government has the opportunity to grow its debt at the same rate as the economy. The result is a dynamic inefficiency which allows a Ponzi game. Finally in this case of exogenous growth, public debt allows a Pareto improvement since it will crowd out the excess capital. King and Ferguson (1993) showed that in models of endogenous growth without learning-by-doing, the steady state is dynamically efficient and Ponzi games are impossible. However, they show that when there is learning-by-doing and when we use the OLG model, a Ponzi game becomes possible again, but this time in a situation of dynamic efficiency. It follows that the Ponzi game will not lead to Pareto improvement because capital is not over but rather under-accumulated. Unlike Tirole (1985) and O'Connell and Zeldes (1988), the over-accumulation is no longer necessary for the existence of a Ponzi game. To clarify, call γ the growth rate, r the interest rate and r^s the social rate of return. Learning-by-doing implies $r^s > r$. Dynamic efficiency requires $r^s > \gamma$. Ponzi game is allowed if $\gamma > r$. In the case $r^s > \gamma > r$ a Ponzi game is possible in situation of under-accumulation and does not produce therefore a Pareto improvement. But the Ponzi game that King and Ferguson got (through learning-by-doing) is a resource for the government that is not used in their model to finance productive public spending.

The assumption of productive public spending has been introduced into a growth model by Barro (1990) under the assumption of financing these expenditures through taxes and a balanced budget. Recently in the line of Greiner and Semmler (2000), an extensive literature has analyzed the long run effects of financing productive public spending by public debt under different fiscal rules. Ghosh and Mourmouras (2004) classify the fiscal rules into four types : *i)* balanced budget rules *ii)* deficit rules *iii)* borrowing rules *iv)* debt rules. For example, the Stability and Growth Pact in the EU imposes a rule of deficit below 3% and a rule of debt below 60%. Germany and the UK impose a rule of borrowing called "golden rule of public finance" according to which borrowing can only finance public investment. Under the golden rule Minea and Villieu (2008), show that financing government spending by debt reduces the growth rate relative to tax financing. Bräuninger (2005) shows that using an AK model, for a debt rule that fixes the ratio debt/GDP and spending rule that fixes the ratio expenditure/GDP, there is a maximum threshold for the deficit ratio and the debt ratio. Yakita (2008) studied in this framework the golden rule and shows that the sustainability threshold depends on the level of public capital. These works are dedicated to the study of the sustainability of fiscal rules and show the negative effects on long run growth of public debt. But this literature does not address the possibility of a Ponzi game that would allow "paying today's public spending with tomorrow's economic growth."

Our first objective is to construct a Ponzi game, which finances productive public spending. As just explained, it is necessary to put in an OLG model where endogenous growth is generated by the learning-by-doing and productive public spending. To do this we first show that it is natural to introduce the learning-by-doing in Barro's model (1990). The purpose of this work is to show the effects on the welfare of productive public spending financed by a Ponzi game. Section 2 gives framework of model. Section 3 gives the steady state. Section 4 shows the possibility of a Ponzi game. Section 5 shows that such a Ponzi game improves the welfare of each generation but deteriorates the welfare of all generations. Section 6 concludes.

2. The model

Consider the accounting government identity where r is the interest rate net of tax :

$$B_{t+1} - B_t + T_t = G_t + r_t B_t \tag{1}$$

We assume that the tax is proportional to GDP. We will express public spending and debt as a proportion of GDP : *i)* $T_t = \tau_t Y_t$, *ii)* $G_t = \varphi_t Y_t$, *iii)* $B_t = b_t Y_t$. Emphasize that these are simple expressions of variables and not fiscal rules. Terms *iii)* is not a debt rule and *ii)* is not a spending rule. In other words we do not assume that (τ, φ, b) are constants as litterature cited in introduction. Dividing each member of the accounting gouvernment identity by B_t, equation (1) can be written as an expression of the growth rate of public debt in period t :

$$\gamma_t^B = \frac{1}{b_t}(\varphi_t - \tau_t) + r_t \tag{2}$$

The only constraint for government is to be solvent in the long run in presence of a positive debt. We assume that the government is solvent as long as the debt is not growing faster than GDP. Thus, the solvency constraint (S) is : $\mathrm{S} \Rightarrow \gamma^B \leq \gamma^Y$. This budget constraint can be more or less binding for government depending on values of the respective interest rate and GDP growth rate.

If $r > \gamma^Y$, the solvency constraint requires the government not to play a Ponzi game (NPG) : $\mathrm{NPG} \Rightarrow \lim_{t\to\infty} B_t(1+r)^{-t} \leq 0 \Leftrightarrow \gamma^B < r$. Indeed, if this ban is not respected, we would have $\lim_{t\to\infty} B_{t+1}/B_t \geq (1+r)$. That would mean in the long run, that debt would grow faster than interest rate, and therefore than the GDP ($\gamma^B \geq r > \gamma^Y$), thereby violating the solvency constraint.

If $r \leq \gamma^Y$, the solvency constraint is satisfied even if the government is playing a Ponzi game. Indeed, suppose a balanced policy $\varphi_t = \tau_t\ \forall t$ (or perpetual deficit $\varphi_t > \tau_t, \forall t$) in presence of positive debt $B_0 > 0$, then from equation (2), we have $\gamma^B = r$ (or $\gamma^B > r$). We are in presence of a Ponzi game, while debt grows at a rate equal to the interest rate (or higher). But the government can play this Ponzi game while remaining solvent since $\gamma^Y \geq \gamma^B = r$ (since $\gamma^Y \geq \gamma^B > r$).

In conclusion, a Ponzi game is possible when the growth rate is higher than the steady state interest rate of long run $\gamma > r$.

In Barro's technology (1990) we introduce, as King and Ferguson's result (1993), learning-by-doing as in Romer (1986). The production function is :

$$Y_t = A{K_t}^{1-\beta}{L_t}^{\beta}{k_t}^{\phi}{g_t}^{\alpha} \tag{3}$$

Dividing by L_t we have $y_t = A{k_t}^{1-\beta}{k_t}^{\phi}{g_t}^{\alpha}$. In addition to two private inputs, capital (K) and labor (L), there are two externalities : On the one hand, public expenditure per capita (g) and, on the other hand, learning-by-doing related to capital stock per capita (k). Appendix (A) showns restrictions on the parameters, especially $1 - \beta + \phi + \alpha = 1$ for endogenous growth. Note that for β given, an increase of α leads to a decrease of ϕ. Any $\beta \in max((\alpha, \varphi), 1)$ can recover the Barro's production function $y_t = A{k_t}^{1-\alpha}{g_t}^{\alpha}$ (see Appendix B). Our production function is the same as Barro, where we exhibit β. This brings learning-by-doing and determines the remuneration of labor which is essential in OLG model.

We assume that capital depreciates in the period at rate δ. The firm's profit after tax is : $(1 - \tau_t) Y_t - w_t L_t - (r_t + \delta)K_t$. In perfect competition, the wage rate equals the marginal product of labor net of taxes and the price of capital equals the marginal product of capital net of tax.

$$w_t = (1 - \tau_t) PmL^{\text{private}} = (1 - \tau_t) \beta A K^{1-\beta} L^{\beta-1} k^{\phi} g^{\alpha} = (1 - \tau_t) \beta y_t \tag{4}$$

$$r_t + \delta = (1 - \tau_t) Pmk^{\text{private}} = (1 - \tau_t)(1 - \beta) A K^{-\beta} L^{\beta} k^{\phi} g^{\alpha} = (1 - \tau_t)(1 - \beta) \frac{y_t}{k_t} \tag{5}$$

By internalizing learning-by-doing and public expenditure policy $g_t = \varphi_t y_t$, the *ex post* production value is (see appendix C) :

$$Y_t = \left(A^{\frac{1}{1-\alpha}} {\varphi_t}^{\frac{\alpha}{1-\alpha}} \right) K_t \tag{6}$$

Using this *ex post* production value, we obtain the *ex post* remuneration values. To simplify, we set $X_t = A^{1/(1-\alpha)}{\varphi_t}^{\alpha/(1-\alpha)}$ and therefore, the interest rate is :

$$r_t = (1 - \tau_t)(1 - \beta) X_t - \delta \tag{7}$$

From the production function *ex post* (6) the social return of capital is :

$$r_t^S = (1 - \tau_t) X_t - \delta \tag{8}$$

Obviously, the social return of capital exceeds its private return because capital generates two externalities. The externality of public spending : the price of capital ignores the fact that its contribution to production will expand the tax base and thus productive public expenditure. The externality of learning-by-doing : the price of capital ignores the fact that its contribution to production generates learning-by-doing.

The consumer behaves as in OLG model as Diamond (1965) without population growth. Their utility function over two periods of life : (y, o) is : $U_t = \ln(c_t^y) + \frac{1}{1+\rho}\ln(c_{t+1}^o)$. Budget constraints when young (y) and old (o), are : $c_t^y + s_t = w_t$ and $c_{t+1}^o = (1 + r_{t+1}) s_t$. By introducing first-order condition in constraints, we find consumption and saving :

$$c_t^y = \frac{1+\rho}{2+\rho} w_t \qquad c_{t+1}^o = \frac{1+r_{t+1}}{2+\rho} w_t \qquad s_t = \frac{1}{2+\rho} w_t \tag{9}$$

3. Steady state

First we suppose that there is no public debt and the government budget is always balanced $\forall t, \varphi_t = \tau_t$. In competitive equilibrium, consumers maximize utility, producers maximize their profits and markets are balanced. The labor market is balanced by assumption $L_t = N_t = N$. By Walras law, it is sufficient that the capital market is balanced. At equilibrium, we have : $K_{t+1} = S_t = Ns_t$. Replacing s_t by (9), w_t by (4) then dividing by K_t, we obtain the competitive balanced growth rate. To be constant at steady state, we must assume $\tau_t = \tau$ and $\varphi_t = \varphi$ we obtain :

$$1+\gamma = \frac{K_{t+1}}{K_t} = \frac{\beta(1-\tau)X}{(2+\rho)} \tag{10}$$

We found the Barro's model growth rate, first increased and then decreased function of $\tau = \varphi$. The maximum is for $\tau = \varphi = \alpha$. In the case where $\alpha = 0$, government expenditure is not productive and growth is generated only by learning-by-doing. Then we have $1+\gamma = \frac{\beta A(1-\tau)}{2+\rho}$. Public expenditure only has a negative effect on growth through taxation as in King and Ferguson (1993). The interest of our model is to introduce the positive effect.

Assume now that the government has a positive debt $B_t > 0$. At equilibrium, we have $B_{t+1} + K_{t+1} = S_t = Ns_t$. In presence of debt, individual savings decreases (when the tax rate is higher in presence of debt) and the aggregate savings must now also invest in government bonds. In presence of public debt, there is a crowding out of capital which reduces growth and, at a given date, the level of capital and production. Growth factor is calculated for capital, GDP and debt ratio in appendix (D). At steady state we have : $\tau_t = \tau_{t+1} = \tau$, $\varphi_t = \varphi_{t+1} = \varphi$, $b_t = b_{t+1} = b$ et $\gamma_t^K = \gamma_t^Y = \gamma_t^B = \gamma$. Growth factors of capital and GDP are :

$$1+\gamma^{Y \text{ and } K} = \frac{\beta\,(1-\tau)\,X}{(2+\rho)\,(1+bX)} \tag{11}$$

It negatively depends on the debt ratio b. If $b = 0$, we find the expression level without debt (10). At steady state, growth factor of Y and K must be egal to the growth factor of debt. Using the equation $1+\gamma^B$ from (2), we have the steady state condition :

$$\frac{\beta\,(1-\tau)\,X}{(2+\rho)\,(1+bX)} = 1 + \frac{1}{b}\,(\varphi-\tau) + (1-\tau)\,(1-\beta)\,X - \delta \tag{12}$$

This steady state condition is satisfied, given the level of the state variable b, only for certain values in the pair of control variables (τ, φ), variables of economic policy. To solve the model, we must consider that one of these two variables is given, and the other one is endogenous. Assume that in the long run, the spending policy is given, φ is exogenous and we solve the equation (12) for τ :

$$\tau = \frac{(2+\rho)\,(1+bX)\,(\varphi + ((1-\beta)\,X + (1-\delta))\,b) - \beta bX}{(2+\rho)\,(1+bX)\,(1+(1-\beta)\,bX) - \beta bX} \tag{13}$$

It is the expression of a curve, which gives tax rates for each level of debt, which provides the steady state, given spending policy set. We see that without debt we find $\tau = \varphi$. Note that with debt τ is not alway greater then φ especially for low level of debt. By replacing the tax rate that ensures the steady state in the expression of the growth rate (11), we express the steady state growth rate only with exogenous parameters :

$$1 + \gamma = \frac{\beta X (1 - \varphi - b(1 - \delta))}{(2 + \rho)(1 + bX)(1 + (1 - \beta) bX) - \beta X b} \tag{14}$$

4. Ponzi game in dynamic efficiency

We show in this section that the economy can be described in the following situation : $r^s > \gamma > r$. In this case, a rational Ponzi game is possible, since $\gamma > r$, with under-accumulation since $r^s > \gamma$, and does not therefore allow Pareto improvement[1]. Our results extend those of Wigger (2009) to a model of endogenous growth and those of King and Ferguson (1993) to a model of endogenous growth with productive government spending. To simplify the presentation, we assume $\delta = 1$, as is usual, in an OLG model in which the period is 30 years.

The economy is dynamically efficient if the capital stock is below the capital stock that maximizes the steady state consumption per capita. It is $c = (1 - \tau)y - (\gamma + \delta)k$ and its maximum is reached for $(1 - \tau)f'(k) = (\gamma + \delta)$ where $f'(k)$ is the derivative of the *ex post* production function (6). So this economy is dynamically efficient if $(1 - \tau) X > \gamma + 1$ so if $r^s \geq \gamma$. From (8) and (11) the condition becomes $1 \geq \beta / ((2 + \rho)(1 + bX))$, which is always true for $b \geq 0$.

Proposition 1 *The economy is always dynamically efficient when debt is positive.*

We have shown that a Ponzi game is possible if $\gamma > r$. By replacing the growth rate and the interest rate by their expressions (11) and (7), then solving successively for the various parameters, we obtain the following proposition :

Proposition 2 *The government can play a Ponzi game if one of the following conditions is satisfied :* $b < \bar{b}$, $\beta > \bar{\beta}$, $\rho < \bar{\rho}$, $\varphi < \bar{\varphi}$, $A < \bar{A}$ *ou* $\alpha < \bar{\alpha}$.

The expression of the thresholds is in Appendix E. Assembling propositions 1 and 2, the following corollary can be established.

Corollary 1 *: The economy is dynamically efficient and the government can play a Ponzi game if : i) the debt is not too large, ii) the wage share is large enough, iii) the time preference rate is not too large, iv) public expenditures are not too large, v) the productivity level is not too large, vi) the productivity of public expenditure is not too large.*

To confirm ideas, we calibrated the model in Appendix F and show that the values of parameters and thresholds that allow a Ponzi game in a dynamically efficient economy are plausible. The condition *i)* is natural, a large debt reduced growth (it does not increase the interest rate in this model) which reduces the possibilities of Ponzi game. The condition

[1]See King and Ferguson (1993) for the difference between the rate of social return and private return in an endogenous growth model and see Wigger (2009) for the difference between gross return and net of tax in an exogenous growth model.

ii) is a characteristic of OLG model in which the young save on wages. A significant share of wages in this model generates a strong accumulation and strong growth. Just as increasing the share of wages reduces the interest rate, it increases the likelihood of a Ponzi game. The condition *iii)* is explained conventionally because a large time preference reduces savings and growth and thus the possibilities of a Ponzi game. Conditions *iv)* and *v)* are explained in the same way : the spending ratio and the productivity level positively involved in $X = A^{1/(1-\alpha)}\varphi^{\alpha/(1-\alpha)}$ which itself comes in the denominator of the growth rate. If φ is too high, growth is weak in Barro's model which reduces the possibility of a Ponzi game. The condition *vi)* seems paradoxical but is the important result of our model. To understand we must remember the results of King and Ferguson (1993). The possibility of a Ponzi game, King and Ferguson (1993) have shown, is related to the importance of learning-by-doing. In our model, the more productive public spending, the less learning-by-doing, when α increases ϕ decreases (see restrictions on exponent in Appendix A), and therefore Ponzi game is less possible. This model provides an opposition between productivity of public spending and the possibility of playing a Ponzi game. This result goes against the idea that it is possible to "pay today's public spending with tomorrow's economic growth."

In the case considered by Diamond (1965), dynamic inefficiency and Ponzi game are mingled and public debt crowds out the excessive capital and unambiguously increases welfare. In our model the effect of debt on welfare is not so obvious. On the one hand, there is no precedent positive effect, since in our model there is always dynamic efficiency. On the otherhand, it was shown that a Ponzi game is possible, and therefore that public spending can be financed by debt and not by tax. A small debt can reduces taxes.

5. Welfare with Ponzi game

We show in this section that at the steady state, the present generation is always interested in a Ponzi game to fund public spending but that the next generation bears the burden systematically. Debt improves the welfare of each generation but deteriorates the welfare of all. The proof is the following: We show that the growth decreases in the presence of debt even if it is a Ponzi game. There is a positive debt that reduces taxes and maximizes the welfare of this generation and it is a Ponzi game. As debt reduces growth, the utility of the next generation decreases. This generation will in turn increase its utility, with debt. It follows that the rational Ponzi game is set up, but is not ultimately a Pareto improvment.

From the equation (14) we calculate debt that maximizes growth : $\frac{\partial\gamma}{\partial b} = 0 \Rightarrow \tilde{b} = -\frac{2(2+\rho)-\beta(3+\rho)}{2X(1-\beta)(2+\rho)}$. As $\rho > -1$ and $\beta < 1$ then $\tilde{b} < 0$. Moreover we compute : $\lim\limits_{b\to-\infty} \frac{\partial\gamma}{\partial b} = -1$, $\lim\limits_{b\to+\infty} \frac{\partial\gamma}{\partial b} = -1$, $\lim\limits_{b\to 0} \frac{\partial\gamma}{\partial b} < 0$. Thus, the economic growth rate is a decreasing function of b for positive values of b. We conclude that growth decrease in the presence of positive debt.

We maximize the utility of agents of the generation born in t. Utility is $U_t = \ln(c_t^y) + \frac{1}{1+\rho}\ln(c_{t+1}^o)$. Substitute the consumption by their expressions (9), wage by (4), assume $k_t = 1$, then replace the tax rate by its steady state expression (13) the utility of generation t can be written as a function of the debt ratio : $U_t = ln\left(\frac{1+\rho}{2+\rho}\right) - \frac{1}{1+\rho}ln\left(2+\rho\right) + \frac{2+\rho}{1+\rho}ln\left(w_t\left[b_t\right]\right) + \frac{1}{1+\rho}ln\left(1+r\left[b_t\right]\right)$. Differentiate this expression with respect to b and we obtain : $\frac{\partial U_t}{\partial b_t} = -\frac{\tau'[b_t]}{(1-\tau[b_t])}\left(\frac{2+\rho}{1+\rho}\frac{\beta}{(1-\beta)} + \frac{1}{1+\rho}\right)$. This derivative vanishes for $\tau'\left[b_t\right] = 0$. That is to say for a debt level that minimizes the tax rate. When $\beta \geq \underline{\beta} = \frac{2+\rho}{3+\rho}$, the positive solution is :

$$b^{*} = \frac{1}{X}\left(-1 + \sqrt{\frac{\beta}{(1-\beta)(2+\rho)}}\right) \geq 0 \tag{15}$$

There is always a positive debt ratio that reduces taxes and maximizes the utility of a generation when $\beta > \underline{\beta}$. It is easily verified that $b^{*} < \bar{b}$ so that debt is necessarily a Ponzi game.

Proposition 3 *The debt ratio b^{*} that reduces taxes and maximizes the utility is necessarily a Ponzi game.*

The question now is, what will the next generation think of this debt ? To analyze this situation, it suffices to consider the consequences of choosing b_t^{*} by generation t on the utility function of generation $t+1$. $U_{t+1} = \ln\left(\frac{1+\rho}{2+\rho}\right) - \frac{1}{1+\rho}\ln(2+\rho) + \frac{2+\rho}{1+\rho}\ln\left(w_{t+1}\left[b_{t+1}, b_t\right]\right) + \frac{1}{1+\rho}\ln\left(1 + r\left[b_{t+1}\right]\right)$. As in steady state, wages grow at rate γ, i.e. $w_{t+1} = (1+\gamma_t)\,w_t)$, any decrease in growth rate, reduces the utility of generation $t+1$. As it was shown that the debt reduces the growth rate, the generation $t+1$ would have preferred that the generation t does not play a Ponzi game.

Of course, following the reasoning that was adopted earlier for generation t, generation $t+1$ will increase its utility also playing a Ponzi game (regardless of the debt made by the generation t for $b_t \in [0, b^{*}]$). The following figure helps to explain this result.

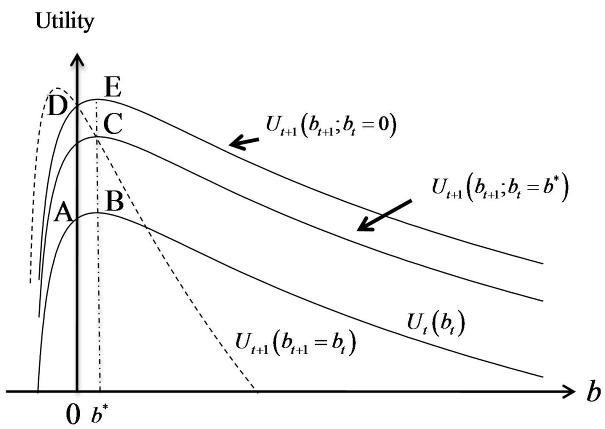

Figure 1: The debt dilemma

Suppose that at the beginning of the period t the debt ratio is zero. By funding public spending only through taxes ($\tau = \varphi$), generation t obtains utility measured at the point A. Generation t can increase its utility by financing a part of public spending by a Ponzi game ($\tau < \varphi$). In this case the utility of this generation t is given at point B.

Regarding the generation $t+1$, the utility level depends on the choice of debt realized by generation t :

If the generation t did not create debt, (curve $U_{t+1}(b_{t+1}; b_t = 0)$) and the generation $t+1$ adopts the same behavior, then, its utility is given in point D. But the generation $t+1$ has an advantage to deviate and to create debt (point E).

If generation t has debt b^* (curve $U_{t+1}(b_{t+1}; b_t = b^*)$), generation $t+1$ maximizes its utility by adopting the same behavior (point C).

We can conclude that generation $t+1$, like generation t increases its utility by creating debt. Generalizing the above reasoning the dominant strategy for each generation is to have debt.

Proposition 4 *Each generation increases its utility by playing a Ponzi game b^*.*

We consider the utility of any generation $(t+1)$ at steady state. The dashed curve on the figure gives the utility of generation $t+1$ when it adopts the same behavior as generation t :$U_{t+1}(b_{t+1} = b_t)$. It represents the utility function for steady state debt. Points D , E and C represent the dilemma of debt. Debt-free steady state, agents would be in D, but this is dominated by the point E where each generation has an interest in debt. All generations are then in steady state at point C. A steady state with debt b^* reduces the utility level of D to C.

Proposition 5 *The Ponzi game reduces the welfare of all generations.*

The explanation of this dilemma is as follows : in this model, labor supply is inelastic, the gross wage is determined by capital resulting of the savings of the previous generation. The only way to have a higher net wage, is to finance government spending by Ponzi game, so pay less tax. A debt that would not be a Ponzi game would involve a tax increase at steady state. A Ponzi game decreases tax rate and increases net wage. The first generation which finances public spending by a Ponzi game increases its utility. But transfer of debt from generation to generation has a cost of crowding out capital, lower income and growth. Thus the chronicle of net wages is lower, with a Ponzi game than without Ponzi game.

6. Conclusion

The Ponzi game improves the welfare of each generation but deteriorates the welfare of all. Each agent pays less tax, but all have less income. As they may decide only the amount of tax they pay and not the stock of capital accumulated through savings from the previous period, they choose to pay less taxes when possible, ie when a Ponzi game is possible.

Ponzi game is possible when share of wages is high and the elasticity of public spending is low, so when elasticity of learning-by-doing is high. This is the free learning-by-doing that allows paying today's public spending with tomorrow's economic growth, not the costly productive government spending.

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A. Restrictions on the exponents

Restrictions on exponents are : *i)* $\beta \in (0,1)$ $i.e$ decreasing returns to scale in the private factors and constants return to scale, *ii)* $1-\beta+\phi+\alpha = 1$ $i.e$ constant returns on accumulated factors, to obtain endogenous growth. This condition implie : $\alpha = \beta - \phi$, $\beta = \alpha + \phi$, $\phi = \beta - \alpha$, *iii)* $\phi \geq 0$ $i.e$ the learning-by-doing externality is positive or null, *iv)* $\alpha \geq 0$ $i.e$ public spending is productive. Deduced from these four restrictions : $Max(\alpha, \phi) \leq \beta \leq 1$, $0 \leq \alpha \leq 1-\phi$, $0 \leq \phi \leq 1-\alpha$.

B. Equivalence with Barro's production function

Indeed suppose $\beta > \alpha$, such that $\beta = \alpha + \varepsilon$ with $\varepsilon > 0$. The production function becomes $y_t = A{k_t}^{1-\beta}{k_t}^{\phi}{g_t}^{\alpha} = A{k_t}^{1-(\alpha+\varepsilon)}{k_t}^{\phi}{g_t}^{\alpha} = A{k_t}^{1-\alpha}{k_t}^{\phi-\varepsilon}{g_t}^{\alpha} = A{k_t}^{1-\alpha}{g_t}^{\alpha}$. Since $\alpha = \beta - \phi$ we have $\beta = \beta - \phi + \varepsilon$ i.e $\varepsilon = \phi$.

C. Internalization of learning-by-doing and public spending policies

$Y_t = A{K_t}^{1-\beta}L^{\beta}(k_t)^{\phi}(\varphi_t y_t)^{\alpha} = A{K_t}^{1-\beta+\phi}L^{\beta-\phi}(\varphi_t y_t)^{\alpha} = A{K_t}^{1-\alpha}(\varphi_t Y_t)^{\alpha}$. Solving for Y_t we obtain 6.

D. Calculation of growth factors

In $B_{t+1}+K_{t+1}=S_t=Ns_t$ replacing s_t by (9) : $B_{t+1}+K_{t+1}=\frac{1}{2+\rho}Nw_t$. Replacing w_t by (4) and dividing by K_t : $\frac{K_{t+1}}{K_t}=\frac{1}{2+\rho}\beta\frac{Y_t}{K_t}(1-\tau_t)-\frac{B_{t+1}}{K_t}$. By replacing the last term[2] : $\left(1+\gamma_t^K\right)=\frac{1}{2+\rho}\beta X_t(1-\tau_t)-b_{t+1}\left(1+\gamma_t^K\right)X_{t+1}$. By solving, we get the capital growth factor :

$$1+\gamma_t^K=\frac{\beta(1-\tau_t)X_t}{(2+\rho)(1+b_{t+1}X_{t+1})}$$

We deduce[3] the growth factor of GDP :

$$1+\gamma_t^Y=\frac{\beta(1-\tau_t)X_{t+1}}{(2+\rho)(1+b_{t+1}X_{t+1})}$$

Finally we obtain[4] the growth factor of debt ratio is :

$$1+\gamma_t^b=\left[\frac{1}{b_t}(\varphi_t-\tau_t)+(1-\tau_t)(1-\beta)X_t-\delta+1\right]\Big/\left[\frac{\beta(1-\tau_t)X_{t+1}}{(2+\rho)(1+b_{t+1}X_{t+1})}\right]$$

E. Thresholds for a Ponzi game

The government can play a Ponzi game when $\gamma>r$ then if :

$$b<\bar{b}=\frac{\beta-(2+\rho)(1-\beta)}{(2+\rho)(1-\beta)X}$$

$$\beta>\bar{\beta}=\frac{(2+\rho)(1+bX)}{(2+\rho)(1+bX)+1}$$

$$\alpha<\bar{\alpha}=\frac{\ln\left(\frac{\beta(\rho+3)-(\rho+2)}{b(1-\beta)(\rho+2)}\right)-\ln(A)}{\ln\left(\frac{\beta(\rho+3)-(\rho+2)}{b(1-\beta)(\rho+2)}\right)+\ln(\varphi)}$$

$$\varphi<\bar{\varphi}=\left(\frac{\beta-(2+\rho)(1-\beta)}{(2+\rho)(1-\beta)bA^{\frac{1}{1-\alpha}}}\right)^{\frac{1-\alpha}{\alpha}}$$

$$\rho<\bar{\rho}=\frac{\beta}{(1-\beta)\left(bA^{\frac{1}{1-\alpha}}\varphi^{\frac{\alpha}{1-\alpha}}+1\right)}-2$$

$$A<\bar{A}=\left(\frac{\beta(2+\rho)\varphi^{\frac{-\alpha}{1-\alpha}}-(2+\rho)}{b(1-\beta)(2+\rho)}\right)^{1-\alpha}$$

F. Calibration

The model is calibrated with the next set of parameters : $A=8;\ \alpha=0.2;\ \rho=0.5;\ b=0.02;\ \varphi=0.2;\ \beta=0.75$. We obtain $r^s=6.2035$, $\gamma=0.831471$, $r=0.800869$. Dividing by 30, the annual rate of social return is 20%, the annual growth rate is 2,8% and the annual interest rate is 2.6%. The economy is dynamically efficiency and the government can play a Ponzi game. With the parameters thus fixed, the thresholds are : $\bar{A}=8.70551$, $\bar{\alpha}=0.321923$, $\bar{\rho}=0.542482$, $\bar{\varphi}=0.305176$, $\bar{\beta}=0.746826$ and $\bar{b}=0.0222285$ so a maximum annual value of debt ratio 66.7%. We note that the values of parameters and thresholds that allow a Ponzi game under dynamic efficiency are plausible.

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[2] $\frac{B_{t+1}}{K_t}=\frac{B_{t+1}}{Y_{t+1}}\frac{Y_{t+1}}{Y_t}\frac{Y_t}{K_t}=b_{t+1}\left(1+\gamma_t^Y\right)\frac{Y_t}{K_t}=b_{t+1}\left(\left(1+\gamma_t^K\right)\frac{X_{t+1}}{X_t}\right)X_t=b_{t+1}\left(1+\gamma_t^K\right)X_{t+1}$

[3] Since $X_t=\frac{Y_t}{K_t}$ we have : $\left(1+\gamma_t^X\right)=\frac{\left(1+\gamma_t^Y\right)}{\left(1+\gamma_t^K\right)}=\frac{X_{t+1}}{X_t}$

[4] Since $b_t=\frac{B_t}{Y_t}$ we have : $\left(1+\gamma_t^b\right)=\frac{\left(1+\gamma_t^B\right)}{\left(1+\gamma_t^Y\right)}$

G. Mathematical notations

Lowercase variables y, k, s, g are per capita.

Y	:	GDP
K	:	Capital stock
L	:	Labor
A	:	Productivity level
B	:	Public debt
G	:	Public spending
S	:	Saving
N	:	Number of agents in each generation
α	:	Elasticity of the externality of public spending
β	:	Elasticity of Labor
ϕ	:	Elasticity of capital externality in the production of learning-by-doing
δ	:	Depreciation rate of capital
ρ	:	Time preference rate
γ	:	Growth rate
τ	:	Tax rate
φ	:	Public spending to GDP ratio
b	:	Public debt to GDP ratio
c_t^y	:	Consumption when young
c_t^o	:	Consumption when old
r	:	Interest rate
r^s	:	Social rate of return on capital
w	:	Wage rate.