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9th January 2014

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Abstract.

We revisit miscellaneous linear differential operators mostly associated with lattice Green functions in arbitrary dimensions, but also Calabi-Yau operators and order-seven operators corresponding to exceptional differential Galois groups. We show that these irreducible operators are not only globally nilpotent, but are such that they are homomorphic to their (formal) adjoints. Considering these operators, or, sometimes, equivalent operators, we show that they are also such that, either their symmetric square or their exterior square, have a rational solution. This is a general result: an irreducible linear differential operator homomorphic to its (formal) adjoint is necessarily such that either its symmetric square, or its exterior square has a rational solution, and this situation corresponds to the occurrence of a special differential Galois group. We thus define the notion of being "Special Geometry" for a linear differential operator if it is irreducible, globally nilpotent, and such that it is homomorphic to its (formal) adjoint. Since many Derived From Geometry n-fold integrals ("Periods") occurring in physics, are seen to be diagonals of rational functions, we address several examples of (minimal order) operators annihilating diagonals of rational functions, and remark that they also seem to be, systematically, associated with irreducible factors homomorphic to their adjoint.

PACS: 05.50.+q, 05.10.-a, 02.30.Hq, 02.30.Gp, 02.40.Xx

AMS Classification scheme numbers: 34M55, 47E05, 81Qxx, 32G34, 34Lxx, 34Mxx, 14Kxx

Key-words: Lattice Green functions, Calabi-Yau ODEs, Ising model operators, differential Galois groups, operators Derived From Geometry, self-adjoint operators, homomorphism or equivalence of differential operators, Special Geometry, globally bounded series, diagonal of rational functions.

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1. Introduction

When one considers all the irreducible factors of the globally nilpotent linear differential operators encountered in the study of n-folds integrals of the Ising class [1] (or the one's displayed by other authors in an enumerative combinatorics framework [2, 3], or in a Calabi-Yau framework [4, 5, 6]), one finds out that their differential Galois groups are not the $SL(N, \mathbb{C})$, or extensions of $SL(N, \mathbb{C})$, groups one could expect generically, but *selected* $SO(N)$, $Sp(N, \mathbb{C})$, G_2, ... differential Galois groups [7].

Along this line it is worth recalling that *globally nilpotent linear differential operators* associated with generic ${}_nF_{n-1}$ *hypergeometric functions with rational parameters*¶, have $SL(N, \mathbb{C})$ (or extensions of $SL(N, \mathbb{C})$) differential Galois groups. For instance the ${}_3F_2$ hypergeometric function

$$ {}_3F_2([\frac{191}{479}, \frac{359}{311}, \frac{503}{89}], [\frac{521}{151}, \frac{401}{67}], x), \qquad (1) $$

has a $SL(3, \mathbb{C})$ differential Galois group‡. In contrast, in the simplest examples, the emergence of "selected" differential Galois groups can be seen very explicitely [10], and understood (from a physicist's viewpoint) as the emergence of some "invariant". As far as the $SO(3, \mathbb{C})$ group is concerned♯, let us consider the *non-Fuchsian*† operator ($\theta = x \cdot d/dx$):

$$ 2\,\theta \cdot (3\,\theta - 2) \cdot (3\,\theta - 4) \; - 9\,x \cdot (2\,\theta + 1), \qquad (2) $$

with the three ${}_1F_2$ hypergeometric solutions

$$ {}_1F_2\Big([\frac{1}{2}], [-\frac{1}{3}, \frac{1}{3}], x\Big), \quad x^{2/3} \cdot {}_1F_2\Big([\frac{7}{6}], [\frac{1}{3}, \frac{5}{3}], x\Big), \quad x^{4/3} \cdot {}_1F_2\Big([\frac{11}{6}], [\frac{5}{3}, \frac{7}{3}], x\Big). $$

If f denotes a solution of this operator (in the above closed form or as a formal solution at the origin or at ∞), one has the following *quadratic relation* $Q(f, f', f") = const.$, where:

$$ Q(X_0, X_1, X_2) \; = \; 9 \cdot (36\,x + 5) \cdot x^2 \cdot X_2^2 - 324 \cdot x^2 \cdot X_2 \cdot X_1 \; - 648\,x^2 \cdot X_2 \cdot X_0 $$
$$ + (81\,x - 5) \cdot X_1^2 \; + 9 \cdot (36\,x - 5) \cdot X_0 \cdot X_1 \; + \; 9 \cdot (36\,x - 5) \cdot X_0^2. $$

The constant depends on the linear combination of solutions used. For instance, with the first ${}_1F_2$ hypergeometric solution one has $Q(f, f', f") = 225/4$, while with the two other ${}_1F_2$ solutions it reads $Q(f, f', f") = 0$. In other words Q is a *first integral*.

The emergence of such "special" differential Galois groups in so many domains of theoretical physics is clearly something *we need to understand better*.

We have provided a large number of linear ODEs on various problems of lattice statistical mechanics, in particular for the magnetic susceptibility of the two-dimensional Ising model [8, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. These linear ODEs factorize into many factors of order ranging from one, to 12 (for $\chi^{(5)}$) and even 23 (for $\chi^{(6)}$). As far as the factors of smallest orders (two, three and four) are concerned, one can verify that *all these linear differential operators are homomorphic to their adjoint*.

¶ Their corresponding linear differential operators are necessarily globally nilpotent [8].

‡ One shows that there are no rational solutions of symmetric powers in degree 2,3,4,6,8,9,12, using an algorithm in M. van Hoeij et al. [9]

♯ This operator is actually homomorphic to its adjoint (see below) with non-trivial order-two intertwiners.

† This operator has an irregular singularity at infinity. At $x = \infty$ the solutions behave like: $t \cdot (1 + 77/72\,t^2 \; + \cdots)$, and $\exp(-2/t)/t \cdot (1 + 13/36\,t + \cdots)$ where $t = \pm 1/\sqrt{x}$.

Furthermore, one remarks experimentally, that their exterior square or symmetric square, either have a *rational solution*, or are of an *order smaller than the order one would expect generically.* Quite often these differential operators are simply *conjugated to their adjoint*, i.e. the intertwiner between the operator and its adjoint, is just an order-zero operator, namely a function. In that case they can easily be recast into *self-adjoint* operators. A large set of linear differential operators *conjugated to their adjoint*, can be found in the very large list of Calabi-Yau order-four operators obtained by Almkvist et al. [5], or displayed by Batyrev and van Straten [4], or some simple order-three operators displayed in a paper by Golyshev [21, 22] (see also Sanabria Malagon [23]).

Throughout this paper we will see examples of *irreducible* operators where these two differential algebra properties occur simultaneously. On the one hand, these operators are *homomorphic to their adjoint*, and on the other hand, their symmetric or exterior square *have a rational*† *solution.* These simultaneous properties correspond to *special differential Galois groups.* In fact, these properties are equivalent‡.

We will, in this paper, have a learn-by-example approach of all these concepts. In this respect, we will display, for pedagogical reasons, a set of enumerative combinatorics examples corresponding to miscellaneous *lattice Green functions* [2, 3, 25, 26, 27] as well as Calabi-Yau examples, together with order-seven operators [29, 30] associated with differential Galois exceptional groups. We will show that these lattice Green operators, Calabi-Yau operators and order-seven operators associated with exceptional groups, are a perfect illustration of differential operators with *selected differential algebra structures*: they are homomorphic to their adjoint, and, either their symmetric, or exterior, powers (most of the time squares) have a *rational solution*, or the previous symmetric, or exterior, powers of some *equivalent operators* have a rational solution. This situation corresponds to the emergence of *selected differential Galois groups*, a situation we could call "Special Geometry". Among the Derived From Geometry n-fold integrals ("Periods") occurring in physics, we have seen that they are quite often *diagonals of rational functions* [18, 19]. We will also address in this paper examples of (minimal order) operators annihilating diagonals of rational functions, and will remark that they also seem to be associated with *irreducible factors homomorphic to their adjoint.*

2. Adjoint of differential operators and invertible homomorphisms of an operator with its adjoint

In the next section, examples of linear differential operators corresponding to lattice Green functions on various lattices are displayed according to their order N and their complexity. We focus on the differential algebra structures of these linear differential operators, in particular with respect to an important "duality" with amounts to performing the *adjoint*, or, more precisely (see 2.1 in [31]), the "formal adjoint" of

† For hyperexponential solutions [24] (command expsols in DEtools), i.e. N-th root of rational solutions, one must consider homomorphisms *up to algebraic extensions.*

‡ In a Tannakian formulation, one could say that the homomorphisms of an operator L_1 with another operator L_2 are isomorphic to the product $Hom(L_1, L_2) \simeq L_1 \otimes L_2^\star$, giving, in the case of the homomorphisms of an operator L with its adjoint $L^\star$, $Hom(L, L^\star) \simeq L \otimes (L^\star)^\star \simeq L \otimes L$, which is isomorphic to the direct sum $L \otimes L \simeq Ext^2(L) \oplus Sym^2(L)$.

the operator (D_x in the whole paper denotes the derivative d/dx):

$$L \,=\, \sum_{n=0}^{N} a_n(x) \cdot D_x^n \quad \longrightarrow \quad adjoint(L) \,=\, (-1)^N \cdot \sum_{n=0}^{N} (-1)^n \cdot D_x^n \cdot a_n(x), \qquad (3)$$

that is:

$$\sum_{n=0}^{N} a_n(x) \cdot \frac{d^n f(x)}{dx^n} \,=\, 0 \quad \longrightarrow \quad (-1)^N \cdot \sum_{n=0}^{N} (-1)^n \cdot \frac{d^n(a_n(x) \cdot f(x))}{dx^n} \,=\, 0. \qquad (4)$$

Since exterior powers play a key role in this paper, it is worth recalling that the adjoint of an N-th order operator L_N is nothing but the $(N-1)$-th exterior power† of this operator, up to a factor that is the Wronskian $W(L_N)$ of the operator:

$$W(L_N) \cdot adjoint(L_N) \,=\, Ext^{N-1}(L_N) \cdot a_N(x) \cdot W(L_N). \qquad (5)$$

2.1. Homomorphisms of an operator with its adjoint

Recall that two operators L and $\tilde{L}$, of the same order, are called homomorphic (see [32, 31]) when there exist two operators T and S of *smaller order* than the one of L and $\tilde{L}$, such that§:

$$\tilde{L} \cdot T \,=\, S \cdot L. \qquad (6)$$

The intertwiner T maps the solutions of L into the solutions of $\tilde{L}$. When T and L have *no common right-factor* (or equivalently when S and $\tilde{L}$ has no common left factor), for example when L is *irreducible*, one can show that this map is bijective. When (6) holds, and T and L have *no common right-factor*, one says that L and $\tilde{L}$ are *equivalent*. Thus, one also has *intertwiners* $\tilde{T}$, $\tilde{S}$ such that

$$L \cdot \tilde{T} \,=\, \tilde{S} \cdot \tilde{L}, \qquad (7)$$

We say that L is *self-adjoint* when $L \,=\, adjoint(L)$. We say that L is *conjugated with its adjoint* when there exists a rational, or N-th root of rational, function f such that $L \cdot f \,=\, f \cdot adjoint(L)$, i.e. $L \cdot f$ is self-adjoint. More generally, a differential operator L is *homomorphic to its adjoint* (in the above sense) when there exists an (intertwiner) operator T (of *order less*♯ *than that of* L) such that¶

$$L \cdot T \,=\, adjoint(T) \cdot adjoint(L). \qquad (8)$$

Again, this means that the operator $L \cdot T$ is self-adjoint.

The typical situation we encounter in physics is such that the differential operators are of a quite large order and factorize into many factors of various orders (see the minimal order operators [11, 13] annihilating the $\chi^{(n)}$'s). For these large order differential operators, we will systematically factorize the operator. *The*

† In maple the exterior power is normalised to be a monic operator (the head polynomial is normalised to 1).

§ The *intertwiner* T is given by the command `Homomorphisms`$(L, \tilde{L})$ of the DEtools package in Maple [33].

♯ Note that the constraint on the order rules out the "tautological" intertwining relation, satisfied by any operator, like $L \cdot adjoint(T) \,=\, T \cdot adjoint(L)$ with $T \,=\, L$.

¶ It is easy to show, in the case of an homomorphism of an operator L with its adjoint, that the intertwiner on the right-hand-side of (8) is necessarily equal to the adjoint of the intertwiner on the left-hand-side. Actually, from the equivalence $L \cdot T \,=\, S \cdot adjoint(L)$, taking adjoint on both sides gives $adjoint(T) \cdot adjoint(L) = L \cdot adjoint(S)$. For irreducible L, the intertwiner is unique, so $S = adjoint(T)$.

interesting concept amounts to seeing if each irreducible factor in the factorization, is homomorphic to its adjoint.

We end this section with two comments. For irreducible L, one deduces, from (6) and (7), the equality

$$L \cdot \tilde{T} \cdot T \;=\; \tilde{S} \cdot S \cdot L, \qquad (9)$$

so the rest of the right division of $\tilde{T} \cdot T$ by L is a constant. When $\tilde{L}$ is the adjoint of L, we will see, in the sequel, that this relation on the intertwiners T and $\tilde{T}$ makes a remarkable "decomposition" of L emerge. The second comment is on the homomorphisms of an operator with its adjoint in the *reducible* case. For two reducible differential operators, L and $\tilde{L}$, of the *same order*, the relation (6) may hold but may not be an equivalence of operators. In the case of a reducible [34] operator having the unique factorization $L = L_n \cdot L_p$, with $n \neq p$, one can show that the homomorphism with the adjoint *just reduces* to a homomorphism of the *right factor* L_p with its adjoint. The corresponding rational solution for the symmetric or exterior square is precisely the rational solution induced by the right factor, since $Sym^2(L_p)$ (resp. $Ext^2(L_p)$) is a right-factor of $Sym^2(L_n \cdot L_p)$ (resp. $Ext^2(L_n \cdot L_p)$).

In the sequel, when studying homomorphism of an operator with its adjoint, we will restrict to *irreducible* operators.

3. Special ODEs from lattice statistical mechanics and enumerative combinatorics: lattice Green functions

We are going to display a set of miscellaneous examples of linear differential operators corresponding to *lattice Green functions* on various lattices. We will denote these lattice Green operators G_n^{latt}, where n is the order of the operators§, and where *latt* refers to the lattice one considers.

3.1. Special lattice Green ODEs: body-centered cubic lattice and simple cubic lattice

One of the simplest example of lattice Green function corresponds to the order-three operator, for the body-centered cubic lattice, given in equation (19) of [3], which reads

$$G_3^{bcc} \;=\; \theta^3 \;-\; 64 \cdot x \cdot (2\,\theta \,+\, 1)^3. \qquad (10)$$

This order-three operator has the ${}_3F_2$ hypergeometric solution

$${}_3F_2\Big([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}], [1, 1]; 512\,x\Big) \;=\; \Big({}_2F_1\Big([\frac{1}{4}, \frac{1}{4}], [1]; 512\,x\Big)\Big)^2, \qquad (11)$$

where the ${}_2F_1$ hypergeometric function is a series with *integer coefficients*:

$${}_2F_1\Big([\frac{1}{4}, \frac{1}{4}], [1]; 512\,x\Big) \;=\; 1 \,+ 32\,x \,+ 6400\,x^2 \,+ 1843200\,x^3 \,+ 623001600\,x^4 \,+\, \cdots$$

The operator (10) is conjugated to its adjoint: $x \cdot adjoint(G_3^{bcc}) \;=\; G_3^{bcc} \cdot x$. The symmetric square of G_3^{bcc} is of *order five* (instead of the order six one could expect generically for order-three operators).

The most well-known example of lattice Green function has been obtained [35] for the *simple cubic lattice*. The lattice Green function corresponds to the order-three

§ Do not expect a simple match between the dimension of the lattice and this order n.

operator (see equation (19) in [3])

$$G_3^{sc} \;=\; \theta^3 \;-\, 2\,x\cdot(10\,\theta^2\,+10\,\theta\,+3)\cdot(2\,\theta\,+\,1) \\ +\,18\,x^2\cdot(2\,\theta\,+\,3)\,(2\,\theta\,+\,2)\,(2\,\theta\,+\,1), \qquad (12)$$

This order-three operator (12), when divided by x on the left, is *exactly self-adjoint*. The symmetric square of G_3^{sc} is of order five (instead of the generic order six).

The solution of (12), which corresponds to a series expansion with *integer coefficients*, is the Hadamard product of $(1-4\,x)^{-1/2}$ with a Heun function, and is also the square of another Heun function which can also be written in terms of ${}_2F_1$ hypergeometric functions with *two possible algebraic pullbacks*:

$$HeunG\Big(9,\,\frac{3}{4},\,\frac{1}{4},\,\frac{3}{4},\,1,\,\frac{1}{2};\;36\,x\Big)^2 \;=\; HeunG\Big(\frac{1}{9},\,\frac{1}{12},\,\frac{1}{4},\,\frac{3}{4},\,1,\,\frac{1}{2};\;4\,x\Big)^2$$

$$=\;(1-4\,x)^{-1/2}\,\star\,HeunG(1/9,1/3,1,1,1,1;x)\;=\;C_\pm^{1/2}\,\cdot\,{}_2F_1\Big([\frac{1}{6},\,\frac{1}{3}],\,[1];\,P_\pm\Big)^2$$

$$=\;1\,+6\,x\,+90\,x^2\,+1860\,x^3\,+44730\,x^4\,+1172556\,x^5\,+\,\cdots \qquad (13)$$

where the algebraic pull-backs $P_\pm$ and algebraic prefactors $C_\pm$ read:

$$P_\pm \;=\; 54\cdot x\cdot\Big(1-27\,x\,+108\,x^2\;\pm(1\,-\,9\,x)\cdot((1-36\,x)\cdot(1-4\,x))^{1/2}\Big),$$

$$C_\pm \;=\; -18\,x\,+\frac{5}{2}\;\pm\frac{3}{2}\cdot((1-36\,x)\cdot(1-4\,x))^{1/2}. \qquad (14)$$

The fact that these selected Heun functions (13) correspond to *modular forms* [20] can be seen on the relation between the two algebraic pullbacks, $y\,=\,P_+$ and $z\,=\,P_-$, namely the genus-zero *modular curve*†:

$$4\cdot\,y^3\,z^3\;-\,12\;y^2\,z^2\cdot(z+y)\;+3\;y\,z\cdot(4\,y^2\,4\,z^2\,-\,127\;y\,z) \\ -\,4\cdot(y\,+z)\cdot(y^2\,+\,z^2\,+83\;y\,z)\;+432\cdot\,y\,z\;\;=\;\;0. \qquad (15)$$

Remark: If one compares two Heun functions with the same singular points and the same critical exponents, which just differ by their *accessory parameter*, namely $HeunG(9,3/4,1/4,3/4,1,1/2,36\,x)$ and¶ $HeunG(9,-3/4,1/4,3/4,1,1/2,36\,x)$, one sees that the first one corresponds to a *modular form* and to series with *integer coefficients*, while the second one *is not even a globally bounded series* [18, 19]. These two Heun functions $HeunG(9,\pm 3/4,1/4,3/4,1,1/2,36\,x)$ are solutions of order-two linear differential operators

$$H_2^{(\pm)} \;=\; \theta^2\;\;-x\cdot(40\,\theta^2+20\,\theta\,\pm3)\;\;+9\cdot x^2\cdot(4\,\theta\,+3)\cdot(4\,\theta\,+1), \qquad (16)$$

which are, both, conjugated to their adjoint:

$$f(x)\cdot\,adjoint(H_2^{(\pm)})\;=\;H_2^{(\pm)}\cdot\,f(x)\quad\text{with:}\quad f(x)\;=\;x\cdot\,((1-36\,x)\cdot(1-4\,x))^{1/2}.$$

† Which is *exactly a rational modular curve* already found for the order-three operator F_3 in [20].
¶ In [35] Joyce adopted the Heun function notation used by Snow [36], which corresponds to a *change of sign in the accessory parameter* q in the Heun function $HeunG(a,q,\alpha,\beta,\gamma,\delta,x)$. Therefore $HeunG(9,3/4,1/4,3/4,1,1/2,*)$ is denoted $F(9,-3/4,1/4,3/4,1,1/2,*)$ in [35]. Unfortunately this old notation, different from the one used, for instance, in Maple, may contribute to some confusion in the literature.

3.2. Special lattice Green ODEs: face-centered cubic lattice

A third order linear differential operator corresponds to the *lattice Green function* of the *face-centered cubic* lattice (see equation (19) in Guttmann's paper [3]):

$$G_3^{fcc} \;=\; \theta^3 \;-2\,x\cdot\theta\cdot(\theta+1)\cdot(2\,\theta+1)\;-\;16\,x^2\cdot(\theta+1)\cdot(5\,\theta^2\;+10\,\theta+6) \\ -\,96\,x^3\cdot(\theta+1)\cdot(\theta+2)\cdot(2\,\theta+3), \qquad (17)$$

where θ is the homogeneous derivative: $\theta \;=\; x\cdot d/dx$. This operator, once divided by x, is *exactly self-adjoint*: $1/x\cdot G_3^{fcc} \;=\; adjoint(1/x\cdot G_3^{fcc})$. The symmetric square of G_3^{fcc} is of order *five* (instead of the order six one could expect for generic order-three operators).

Let us introduce, instead of G_3^{fcc}, the equivalent operator $\tilde{G}_3^{fcc}$ such that†

$$S_1^{fcc}\cdot G_3^{fcc} \quad=\quad \tilde{G}_3^{fcc}\cdot\; D_x. \qquad (18)$$

where the order-one intertwiner S_1^{fcc} reads up to a factor

$$D_x \;\;-\frac{d\ln(\rho(x))}{dx}, \qquad (19)$$

where the Wronskian $\rho(x)$ is a rational function:

$$\rho(x) \;\;=\;\; \frac{6\,x\,+\,1}{x\,\cdot\,(4\,x\,+1)^2\,\cdot\,(12\,x\,-1)}. \qquad (20)$$

We find that the *symmetric square* of the equivalent operator $\tilde{G}_3^{fcc}$ has a *rational solution* $r(x)$:

$$r(x) \;\;=\;\; \frac{1}{x^2\,\cdot\,(4\,x\,+1)^2\,(12\,x\,-1)}. \qquad (21)$$

More precisely, the *symmetric square* of the equivalent operator $\tilde{G}_3^{fcc}$ is the *direct sum*§ of an order-one operator and an order-five operator:

$$Sym^2(\tilde{G}_3^{fcc}) \;=\; M_1\,\oplus\,M_5 \qquad \text{where:} \qquad M_1 \;=\; D_x\,-\,\frac{d\ln(r(x))}{dx}. \qquad (22)$$

The Wronskian of G_3^{fcc} is the square root of a rational function. The differential Galois group is not the generic $SL(3,\,\mathbb{C})$ one could expect for a generic order-three operator, but is equal to the orthogonal group $O(3,\,\mathbb{C})$: the rational solution (21) of $Sym^2(\tilde{G}_3^{fcc})$, comes from an invariant of degree 2 for the differential Galois group.

In fact the operator (17) happens to be the *symmetric square* of an order-two operator¶:

$$\theta^2 \;\;-\,2\,x\cdot\theta\cdot(4\,\theta+1)\;\;-24\,x^2\cdot\,(\theta+1)\,(2\,\theta+1). \qquad (23)$$

From that last remark, one immediately deduces that the differential Galois group must be the differential Galois group of an order-two operator, generically $SL(2,\,\mathbb{C})$. Indeed, $O(3,\,\mathbb{C})$ is a symmetric square of $SL(2,\,\mathbb{C})$ (see [37]). It is shown in [37] that a third order operator has a symmetric square of order five (instead of six) if, and only if, it is the symmetric square of a second order operator.

† The operator $\tilde{G}_3^{fcc}$ can be obtained from the rightdivision of the $LCLM(G_3^{fcc},\,D_x)$ by D_x, the operator S_1^{fcc} being obtained from the rightdivision of the $LCLM(G_3^{fcc},\,D_x)$ by G_3^{fcc}.

§ A consequence of the fact that, in this case, the differential Galois group is *reductive* so its representations are *semi-simple*.

¶ Conjugated to its adjoint by the function $(1-12\,x)^{1/2}\cdot\,x$

3.3. Special lattice Green ODEs: diamond lattice

Another example can be found in Guttmann and Prellberg [3, 25], and corresponds to an order-three operator which has the following ${}_3F_2$ solution

$$\frac{1}{(4\,-\,x^2)^3}\,\cdot\,\,{}_3F_2\Big([\frac{1}{3},\,\frac{1}{2},\,\frac{2}{3}],\,[1,\,1],\,\,\frac{27\,x^4}{(4\,-\,x^2)^3}\Big), \qquad (24)$$

associated with the *Green function of the diamond lattice*. This order-three linear differential operator reads:

$$\begin{aligned} G_3^{diam} \quad = \quad & 64\cdot\theta^3\;-\;16\,x^2\cdot(7\,\theta^3\,+27\,\theta^2\,+42\,\theta\,+24) \\ & +\;12\,x^4\cdot(5\,\theta^3\,+42\,\theta^2\,+124\,\theta\,+128) \\ & -\;x^6\cdot(13\,\theta^3\,+171\,\theta^2\,+762\,\theta\,+1152)\;\;+\;\;x^8\cdot(\theta\,+6)^3, \end{aligned} \qquad (25)$$

which can be seen to be conjugated to its adjoint:

$$\frac{(x^2-4)^2}{x}\cdot G_3^{diam} \quad = \quad adjoint(G_3^{diam})\cdot\frac{(x^2-4)^2}{x}.$$

The symmetric square of G_3^{diam} is of order *five* (instead of the order six one could expect for generic order-three operators).

Let us introduce, instead of G_3^{diam}, the equivalent operator $\tilde{G}_3^{diam}$

$$S_1^{diam}\cdot\,G_3^{diam}\quad=\quad\tilde{G}_3^{diam}\cdot\;\;D_x. \qquad (26)$$

where the order-one intertwiner S_1^{diam} reads

$$S_1^{diam}\quad=\quad-\,\frac{(x^2-4)^2}{x}\cdot\,r(x)\cdot\,\Big(D_x\;-\frac{1}{\rho(x)}\cdot\frac{d\rho(x)}{dx}\Big), \qquad (27)$$

where $r(x)$ and $\rho(x)$ are rational functions:

$$r(x)\;\;=\;\;\frac{1}{(x-2)^5\,(x-1)\,(x+1)\,(x+2)^5\cdot\,x^2}, \qquad \text{and:} \qquad (28)$$

$$\rho(x)\;\;=\;\;(3\,x^3\,-8\,x+4)\,(3\,x^3\,-8\,x\,-4)\cdot\,x^2. \qquad (29)$$

Again, the *symmetric square* of the equivalent operator $\tilde{G}_3^{diam}$ has the *rational solution* $r(x)$.

The Wronskian of (25) is the square root of a rational function. The differential Galois group is again $O(3,\,\mathbb{C})$.

As in the previous example, G_3^{diam} is the *symmetric square* of an order-two operator:

$$\begin{aligned} & 16\cdot\theta^2\;\,-12\,x^2\cdot\,(2\,\theta^2\,+3\,\theta\,+2) \\ & \qquad\qquad +3\,x^4\cdot\,(3\,\theta^2\,+11\,\theta\,+12)\;-\,x^6\cdot\,(\theta\,+3)^2. \end{aligned} \qquad (30)$$

3.4. Order-three operators conjugated to their adjoint

In fact, these previous results, for the bcc, sc, fcc, diamond lattices, can be seen as the consequence of the following general result on *order-three* linear differential operators (without any loss of generality we restrict to monic operators)

$$L_3\;\;=\;\;D_x^3\;\,+\,a_2(x)\cdot\,D_x^2\;\,+\,a_1(x)\cdot\,D_x\;\,+\,a_0(x). \qquad (31)$$

Any (monic) order-three operator which is conjugated to its adjoint, namely $L_3 \cdot f(x) = f(x) \cdot adjoint(L_3)$, is the symmetric square of an order-two operator

$$L_2 = D_x^2 + b_1(x) \cdot D_x + b_0(x), \qquad \text{with:} \qquad b_1(x) = -\frac{1}{2}\frac{1}{f(x)}\frac{df(x)}{dx}, \tag{32}$$

$$\text{where:} \qquad b_0(x) = \frac{a_1(x)}{4} + \frac{1}{8}\frac{1}{f(x)} \cdot \frac{d^2f(x)}{dx^2} - \frac{1}{4} \cdot \Bigl(\frac{1}{f(x)} \cdot \frac{df(x)}{dx}\Bigr)^2.$$

Note that one necessarily has $a_2(x) = 3\, b_1(x)$. The Wronskian of L_3 is necessarily equal to $f(x)^{3/2}$, and the order-two operator (32) is conjugated to its adjoint by a function:

$$f(x)^{1/2} \cdot adjoint(L_2) = L_2 \cdot f(x)^{1/2}. \tag{33}$$

The symmetric square of such an order-three operator L_3, conjugated to its adjoint, is of order *five* (in contrast to order six for symmetric squares of generic order-three operators).

3.5. Special lattice Green ODEs: 4D face-centered cubic lattice

A slightly more involved example, corresponding to the *four-dimensional face-centered cubic lattice Green function*, can be found in paragraph 2.5 of Guttmann's paper [3] (it is also ODE number 366 in the list of Almkvist et al. [5]). This order-four linear differential operator

$$\begin{aligned}
G_4^{4Dfcc} \;=\;& \theta^4 + x \cdot (39 \cdot \theta^4 - 30 \cdot \theta^3 - 19 \cdot \theta^2 - 4\,\theta) \\
&+ 2\,x^2 \cdot (16 \cdot \theta^4 - 1070 \cdot \theta^3 - 1057 \cdot \theta^2 - 676\,\theta - 192) \\
&- 36\,x^3 \cdot (171 \cdot \theta^3 + 566 \cdot \theta^2 + 600\,\theta + 316) \cdot (3\,\theta + 2) \\
&- 2^5\,3^3\,x^4 \cdot (384 \cdot \theta^4 + 1542 \cdot \theta^3 + 2635 \cdot \theta^2 + 2173\,\theta + 702) \\
&- 2^6\,3^3\,x^5 \cdot (1393 \cdot \theta^3 + 5571 \cdot \theta^2 + 8378\,\theta + 4584) \cdot (\theta + 1) \\
&- 2^{10}\,3^5\,x^6 \cdot (31 \cdot \theta^2 + 105\,\theta + 98) \cdot (\theta + 1) \cdot (\theta + 2) \\
&- 2^{12}\,3^7\,x^7 \cdot (\theta + 1) \cdot (\theta + 2)^2 \cdot (\theta + 3) \\
=\;& x^4 \cdot (1 + 3\,x)\,(1 + 4\,x)\,(1 + 8\,x)(1 + 12\,x)\,(1 + 18\,x)^2\,(1 - 24\,x) \cdot D_x^4 + \cdots
\end{aligned} \tag{34}$$

can be seen to be conjugated to its adjoint by a function f^{4Dfcc}:

$$G_4^{4Dfcc} \cdot f^{4Dfcc} = f^{4Dfcc} \cdot adjoint(G_4^{4Dfcc}), \quad \text{with:} \quad f^{4Dfcc} = x \cdot (1 + 18\,x)^3.$$

The *exterior square* of operator (34) is an irreducible *order-five* operator (not order-six as could be expected): one easily checks that the "Calabi-Yau condition" (see [6] and (94) below) is actually satisfied for operator (34). If one considers an operator $\tilde{G}_4^{4Dfcc}$, non-trivially homomorphic [32, 31] to G_4^{4Dfcc}, its *exterior square* is, now, an operator of (the generic) order six, and it has a *rational solution.* For instance, if we consider the operator $\tilde{G}_4^{4Dfcc}$ equivalent to G_4^{4Dfcc}

$$S_1^{4Dfcc} \cdot G_4^{4Dfcc} = \tilde{G}_4^{4Dfcc} \cdot D_x. \tag{35}$$

where

$$S_1^{4Dfcc} = -\frac{r(x)}{(1 + 18\,x)^3 \cdot x} \cdot \Bigl(D_x - \frac{d\ln(\rho(x))}{dx}\Bigr), \qquad \text{with} \tag{36}$$

$$r(x) = \frac{18\,x + 1}{x^3 \cdot (3\,x + 1)\,(4\,x + 1)\,(8\,x + 1)\,(12\,x + 1)\,(24\,x - 1)}, \qquad \text{and}$$

$$\rho(x) = (1119744\,x^5 + 508032\,x^4 + 82512\,x^3 + 6318\,x^2 + 237\,x + 4) \cdot x,$$

we find that the *exterior square* of $\tilde{G}_4^{4Dfcc}$ has the *rational solution* $r(x)$.

This is a situation we will encounter many times: switching to an equivalent operator "desingularizes" the drop of order of the exterior (resp. symmetric) n-th power of an operator, into a situation of emergence of a rational solution for that n-th power.

The Wronskian of G_4^{4Dfcc} is a rational function. As the exterior square of $\tilde{G}_4^{4Dfcc}$ has a *rational solution*, the differential Galois group is included in the *symplectic group* $Sp(4, \mathbb{C})$. Moreover, its symmetric square being irreducible, theorems A.5 et A.7 of Beukers et al. [38] show that the differential Galois group is exactly $Sp(4, \mathbb{C})$.

3.6. Another version of the order four operator for 4D fcc lattice

Another version¶ of G_4^{4Dfcc}, can be found in the unpublished paper of D. Broadhurst (see equation (68) in [39]) and corresponds to the order-four operator

$$G_4^{4D} \quad = \quad (2\,\theta)^4 \; + \sum_{j=1}^{6} (-1)^j \cdot P_j(2\,\theta + j) \quad = \quad h_4 \cdot D_x^4 \; + \; \cdots\,, \tag{37}$$

with

$$P_1(u) \;=\; 105\,u^4 + 166\,u^2 + 17, \qquad P_2(u) \;=\; 2 \cdot (2095\,u^4 + 2912\,u^2 + 432),$$
$$P_3(u) \;=\; 72 \cdot (1155\,u^4 - 892\,u^2 + 577), \quad P_4(u) \;=\; 864 \cdot (1011\,u^4 - 5059\,u^2 + 4900),$$
$$P_5(u) \;=\; 75600 \cdot (u^2 - 9)\,(61\,u^2 - 145), \quad P_6(u) \;=\; 9525600 \cdot (u^2 - 4)\,(u^2 - 16),$$

and where the head polynomial h_4 reads:

$$h_4 \quad = \quad 16\,x^4 \cdot (1-6\,x)\,(1-10\,x)\,(1-14\,x)\,(1-15\,x)\,(1-18\,x)\,(1-42\,x).$$

Operator G_4^{4D} is a globally nilpotent operator, as can be seen on the Jordan form reduction of its p-curvature [8, 40].

This irreducible order-four operator G_4^{4D} is conjugated to its adjoint (or exactly self-adjoint with a different normalization):

$$G_4^{4D} \cdot x \quad = \quad x \cdot adjoint(G_4^{4D}). \tag{38}$$

Its *exterior square* is an irreducible *order-five* linear differential operator (not order-six as could be expected): the "order-five Calabi-Yau condition" (see (94) below) is satisfied for (37). Similarly to (35) and (44), one can switch, by operator equivalence, to an operator $\tilde{G}_4^{4D}$ such that its *exterior square* has a *rational solution*:

$$S_1^{4Dfcc} \cdot G_4^{4D} \quad = \quad \tilde{G}_4^{4D} \cdot \; D_x. \tag{39}$$

where the order-one intertwiner S_1^{4D} reads

$$S_1^{4D} \quad = \quad -\,\frac{r(x)}{x} \cdot \Big(D_x \; - \frac{d\ln(\rho(x))}{dx}\Big), \tag{40}$$

with

$$\rho(x) \;\; = \;\; (63504000\,x^5 - 17388000\,x^4 + 1644948\,x^3 - 64578\,x^2 + 950\,x - 3) \cdot x,$$

and where $r(x)$ is the rational function:

$$r(x) \;\; = \;\; \frac{1}{x^3 \cdot \; (6\,x-1)\,(10\,x-1)\;(14\,x-1)\,(15\,x-1)\;(18\,x-1)\,(42\,x-1)}. \tag{41}$$

¶ Corresponding to a change of variable: $F(x) \;\rightarrow\; F(x/(1-18\,x))/(1-18\,x)$.

As above, the *exterior square* of the equivalent operator $\tilde{G}_4^{4D}$ has (obviously) the *rational solution* $r(x)$: the differential Galois group of G_4^{4D} is not $SL(4, \mathbb{C})$ but is included in the *symplectic group* $Sp(4, \mathbb{C})$.

One can get, directly, a rational solution if one switches to the *linear differential system* associated with G_4^{4D}, calculates† the *exterior square of that system* and seeks for the rational solution of that exterior square system.

One gets, that way, the rational solution of the exterior square of the *companion system* of G_4^{4D}:

$$\Big[0, \quad 0, \quad r(x), \quad -r(x), \quad x^2 \cdot q_5 \cdot r(x)^2, \, -x \cdot q_6 \cdot r(x)^2\Big], \qquad (42)$$

where:

$$q_5 \, = \, 85730400\, x^6 \, - 36892800\, x^5 \, + 6114528\, x^4 \, - 498960\, x^3 \, + 20950\, x^2 \, - 420\, x \, + 3,$$
$$q_6 \, = \, 190512000\, x^6 - 72198000\, x^5 + 10262808\, x^4 - 690804\, x^3 + 22406\, x^2 - 304\, x + 1.$$

At first sight, for a physicist, performing calculations on a linear differential operator, like $\tilde{G}_4^{4D}$, and finding that its exterior square has the *rational solution* (41), seems simpler, and more natural, than introducing the *companion system.* As far as practical calculations are concerned, the calculations on the linear differential systems turn out to be *drastically more efficient*, and allow to handle symmetric and exterior powers constructions on larger examples.

3.7. Special lattice Green ODEs: 5D staircase polygons

Another example of Guttmann and Prellberg [3, 25], corresponding to the generating function of the *five-dimensional staircase polygons*, is the order-four operator

$$G_4^{5D} \, = \, \theta^4 \, - x \cdot (35\, \theta^4 \, + 70\, \theta^3 \, + 63\, \theta^2 \, + 28\, \theta \, + 5) \qquad (43)$$
$$+ \, x^2 \cdot (259\, \theta^2 \, + 518\, \theta \, + \, 285) \cdot (\theta \, + \, 1)^2 \, - \, 225\, x^3 \cdot (\theta \, + \, 1)^2 \cdot (\theta \, + \, 2)^2$$
$$= \, x^4 \cdot (1 \, + 35\, x \, + 259\, x^2 \, - 225\, x^3) \cdot D_x^4 \, + \, \cdots$$

which can be seen to be conjugated to its adjoint:

$$G_4^{5D} \cdot x \, = \, x \cdot adjoint(G_4^{5D}).$$

The exterior square operator of the order-four operator (43) is an irreducible *order-five* operator (not order-six as could be expected): the "order-five Calabi-Yau condition" (see (94) below) is satisfied for (43). Let us introduce, instead of G_4^{5D}, the equivalent operator $\tilde{G}_4^{5D}$ corresponding to the intertwining relation

$$S_1^{5D} \cdot G_4^{5D} \, = \, \tilde{G}_4^{5D} \cdot \, D_x, \qquad (44)$$

where the order-one intertwiner S_1^{5D} reads

$$S_1^{5D} \, = \, - \frac{r(x)}{x} \cdot \Big(D_x \, - \frac{d \ln((60\, x \, + 1)\, (3\, x \, - 1)\, x)}{dx}\Big), \qquad (45)$$

and where $r(x)$ is the rational function:

$$r(x) \, = \, \frac{1}{(225\, x^3 \, - 259\, x^2 \, - 35\, x \, - 1) \cdot \, x^3}. \qquad (46)$$

† In order to do these calculations download the three Maple Tools files TensorConstructions.m and IntegrableConnections.m in the web page [41]. Using DEtools, you will need to use, on the order-six operator G_4^{4D} written in a monic way, the command companion-system, then the command exterior-power-system(,2), and, finally, the command RationalSolutions([],[x]).

We find, again, that the exterior square of the equivalent operator $\tilde{G}_4^{5D}$ has the *rational solution* $r(x)$. The Wronskian of G_4^{5D} is a rational function. The differential Galois group is, again (see A.5 et A.7 in appendix A of [38]), the *symplectic group* $Sp(4, \mathbb{C})$.

3.8. Order-six operator by Broadhurst and Koutschan: the lattice Green function of the five-dimensional fcc lattice

A more involved example of order-six, can be found in Koutschan's paper [27] and in an unpublished paper of D. Broadhurst (see equation (74) in [39]) and corresponds to a *five-dimensional fcc lattice*

$$G_6^{5Dfcc} \quad = \quad 3^4 \cdot \theta^5 \cdot (\theta \, - 1) \; + \sum_{j=1}^{12} x^j \cdot Q_j(\theta) \quad = \quad h_6 \cdot D_x^6 \; + \; \cdots, \qquad (47)$$

where the polynomials Q_j are degree-six polynomials with integer coefficients, and where the head polynomial h_6 reads:

$$h_6 \; = \; x^6 \cdot \lambda(x) \cdot p_6,$$

$$\text{with:} \qquad \lambda(x) \; = \; (1-4\,x)\,(1-8\,x)\,(1+16\,x)\,(1-16\,x)\,(1-48\,x)\,(3-16\,x),$$

$$\text{and:} \qquad p_6 \; = \; 916586496\,x^6 \; - 571981824\,x^5 \; + 67242496\,x^4 \; - 8372096\,x^3 \; + 315096\,x^2 \; - 6840\,x \; + 27. \qquad (48)$$

This order-six linear differential operator has, at the origin $x = 0$, *two independent analytic solutions* (no logarithms, it is *not* MUM† [19]). One can build, from these two solutions, a one-parameter family of analytic solutions:

$$1 \; + 8 \cdot x \cdot c \; + \frac{8}{3} \cdot (41 \cdot c - 2) \cdot x^2 \; + \frac{32}{27} \cdot (1933 \cdot c - 286) \cdot x^3 \; + \frac{8}{27} \cdot (183136 \cdot c - 25537) \cdot x^4 \; + \frac{256}{2025} \cdot (12082067 \cdot c - 1788992) \cdot x^5 \; + \cdots$$

which, for $c = 1$, (and only this value) becomes a series with *integer*‡ coefficients:

$$1 \; + 8 \cdot x \; + 104 \cdot x^2 \; + 1952 \cdot x^3 \; + 46696 \cdot x^4 \; + 1301248 \cdot x^5 \; + 40047584 \cdot x^6 \; + 1319992192 \cdot x^7 \; + 45737941096 \cdot x^8 \; + 1646328483008 \cdot x^9 \; + \cdots$$

The question of the integrality of such D-finite series, emerging from physics, is addressed in previous papers [18, 19].

Remark: The other unique independent no-log series starting with x reads:

$$z_0(x) \; = \; x \; + \frac{41}{3} \cdot x^2 \; + \frac{7732}{27} \cdot x^3 \; + \frac{183136}{27} \cdot x^4 \; + \frac{386626144}{2025} \cdot x^5 \; + \frac{106836145888}{18225} \cdot x^6 \; + \frac{172725353100416}{893025} \cdot x^7 \; + \cdots$$

It is *not a globally bounded* series [18, 19], i.e. it is *not* a series that can be recast into a series with integer coefficients after a rescaling of the variable.

† MUM means maximally unipotent monodromy [3, 20, 43].
‡ The integrality of these coefficients has been checked with 2000 coefficients, and the coefficients $c_{c\cdot 10000} \cdot x^{c \cdot 10000}$ coefficients, for $c = 1, 2, 3, 4$, have also been seen to be integers.

This order-six linear differential operator is *globally nilpotent* [8, 40], the Jordan reduction of its p-curvature [8, 40] reading:

$$J_6 \; = \; \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (49)$$

We found that the order-six operator G_6^{5Dfcc} is *non-trivially homomorphic to its adjoint*, with a simple *order-one* intertwiner

$$G_6^{5Dfcc} \cdot T_1^{5Dfcc} \; = \; adjoint(T_1^{5Dfcc}) \cdot \, adjoint(G_6^{5Dfcc}), \qquad (50)$$

with:

$$T_1^{5Dfcc} \; = \; x^2 \cdot p_2 \cdot p_6 \cdot \Big(D_x \, - \, \frac{1}{2} \cdot \frac{d \ln(R(x))}{dx} \Big), \qquad \text{where}$$

$$R(x) \, = \, \frac{p_2^5}{x^4 \cdot p_6^4} \qquad \text{with} \qquad p_2 \; = \; 1152\, x^2 \, - 56\, x \, - 3. \qquad (51)$$

Introducing

$$\rho(x) \; = \; \frac{{p_2}^6}{{p_6}^3 \cdot \, x^2}, \qquad (52)$$

the previous order-one intertwiner T_1^{5Dfcc}, can be seen as the product of the rational function $\rho(x)$, and of a *self-adjoint* order-one operator Y_1^s:

$$T_1^{5Dfcc} \, = \, \rho(x) \cdot Y_1^s, \qquad Y_1^s \; = \; \frac{1}{R(x)} \cdot \Big(D_x \, - \, \frac{1}{2} \cdot \frac{d \ln(R(x))}{dx} \Big). \qquad (53)$$

The other intertwining relation is a bit more involved since the intertwiner is an *order-five* linear differential operator S_5^{5Dfcc}

$$adjoint(S_5^{5Dfcc}) \cdot G_6^{5Dfcc} \; = \; adjoint(G_6^{5Dfcc}) \cdot S_5^{5Dfcc}, \qquad (54)$$

where

$$S_5^{5Dfcc} \; = \; \frac{x^2 \, \cdot \, \lambda(x) \cdot \, p_2^5}{p_6^3} \cdot \Big(D_x^5 \; - \frac{1}{2} \cdot \frac{d \ln(\mu(x))}{dx} \cdot D_x^4 \; + \; \cdots \Big)$$

with $\lambda(x)$ as above in (48), and:

$$\mu(x) \; = \; - \frac{p_2^5}{\lambda(x)^5 \, \cdot \, x^{20}}.$$

Quite remarkably, introducing the *same* function $\rho(x)$ as for T_1^{5Dfcc} (see (52)), the previous order-five intertwiner S_5^{5Dfcc}, can be seen as the product $S_5^{5Dfcc} = \rho(x) \cdot Y_5^s$, of the rational function $\rho(x)$ (see (52)) and of a *self-adjoint* order-five operator

$$Y_5^s \; = \; \frac{x^4 \, \cdot \, \lambda(x)}{p_2} \cdot \Big(D_x^5 \; - \frac{1}{2} \cdot \frac{d \ln(\mu(x))}{dx} \cdot D_x^4 \; + \; \cdots \Big). \qquad (55)$$

The self-adjoint order-five irreducible operator Y_5^s has a solution analytic at $x = 0$, (the other solution have log terms), which has the following expansion

$$1 \;+8\,x\;+102\,x^2\;+\frac{487192}{243}\,x^3\;+\frac{86597215}{1944}\,x^4\;+\frac{22841991292}{16875}\,x^5 \\ +\frac{1874527149707741}{49207500}\,x^6\;+\frac{403024701445683311141}{29536801875}\,x^7\;+\;\cdots \qquad (56)$$

This solution-series is *not globally bounded* [18, 19]. The examination of the formal series solutions at $x = 0$ corresponds to a MUM structure.

The self-adjoint order-five irreducible operator Y_5^s is such that its *symmetric square is of order 14 instead of the order 15 expected generically* (its exterior square is of order 10 as it should, with no rational solution).

The Wronskian of this order-six linear differential operator G_6^{5Dfcc} is the square root of a rational function:

$$W(G_6^{5Dfcc}) \;=\; \Big(\frac{p_6^2}{x^{28}\cdot\lambda(x)^7}\Big)^{1/2}.$$

The previous homomorphisms of the order-six operator G_6^{5Dfcc} with its adjoint, namely (50) and (54), can be simply rewritten in terms of the self-adjoint operators Y_1^s and Y_5^s:

$$G_6^{5Dfcc}\cdot\rho(x)\cdot Y_1^s \;=\; Y_1^s\cdot\rho(x)\cdot adjoint(G_6^{5Dfcc}), \qquad (57)$$

$$Y_5^s\cdot\rho(x)\cdot G_6^{5Dfcc} \;=\; adjoint(G_6^{5Dfcc})\cdot\rho(x)\cdot Y_5^s. \qquad (58)$$

From these two intertwining relations it is straightforward§ to see that an operator of the form

$$\Omega_6 \;=\; Y_1^s\cdot\rho(x)\cdot Y_5^s \;+\; \frac{\alpha}{\rho(x)}, \qquad (59)$$

satisfies the *same* intertwining relations (57) and (58), as G_6^{5Dfcc}. It is, in fact, a straightforward calculation to see that the order-six operator G_6^{5Dfcc} is *actually of the form* (59) with $\alpha = -192$:

$$G_6^{5Dfcc} \;=\; Y_1^s\cdot\rho(x)\cdot Y_5^s \;-\; \frac{192}{\rho(x)}. \qquad (60)$$

Recalling section 2, and, more precisely, the fact that the right division of $\tilde{T}\cdot T$ by L is a constant (see (9)), one can rewrite (60) as:

$$\rho(x)\cdot Y_1^s\cdot\rho(x)\cdot Y_5^s \;=\; 192\;+\rho(x)\cdot G_6^{5Dfcc}. \qquad (61)$$

In other words, the two intertwiners $\rho(x)\cdot Y_1^s$ and $\rho(x)\cdot Y_5^s$ are *inverse of each other modulo the operator* $\rho(x)\cdot G_6^{5Dfcc}$.

The operator Y_5^s is *not globally nilpotent* [8], however we see the emergence of some structure: the Jordan form of the p-curvature of this self-adjoint operator Y_5^s reads

$$\left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & R_1^p & 0 & R_2^p & 0 \end{array}\right], \qquad (62)$$

§ Using the identity $adjoint(\Omega\,+\,f(x)) \,=\, adjoint(\Omega) + f(x)$ valid for *any even order operator* Ω, and for *any* function $f(x)$.

its characteristic polynomial reading $x \cdot (x^4 - R_2^p \cdot x^2 - R_1^p)$.

The *exterior square* $Ext^2(G_6^{5Dfcc})$ is an order-fifteen linear differential operator which *does not have a rational solution* (or a hyperexponentional solution, see chapter 4 of [31] and [42]), thus *excluding a symplectic structure* with a $Sp(6, \mathbb{C})$ differential Galois group.

In contrast, its *symmetric square* $Sym^2(G_6^{5Dfcc})$, which does not have a rational solution, is of *order* 20 *instead of the generic order* 21. In fact, the *associated differential system does have a rational solution*. Again, we see that the operator representation is not "intrinsic" enough, differential systems are more "intrinsic" from a practical viewpoint. The emergence (for the system) of a rational solution for the symmetric square means that the differential Galois group is included¶ in the orthogonal group $O(6, \mathbb{C})$.

From that viewpoint, the order-six operator G_6^{5Dfcc} seems to contradict an "experimental" principle‡ that orthogonal groups occur from odd order operators, and symplectic groups occur from even order operators. In fact the exceptional character of this even order operator comes from this decomposition (60) in terms of *odd order* intertwiners (see (57) and (58)).

The log structure of the solutions is exactly the same as the one of a symmetric square of an order-three operator, $Sym^2(L_3)$, which might suggest that the differential Galois group would be the differential Galois group of a MUM order-three operator (generically $SL(3, \mathbb{C})$).

3.8.1. System representation of G_6^{5Dfcc}

The following calculations are performed using the "system representation" as in the previous section. One gets§ the *rational solution* of the *symmetric square* of the companion system:

$$[c_1, \quad c_2, \quad c_3, \quad \cdots, \quad c_{21}] \quad = \tag{63}$$

$$= \Big[0,\ 0,\ 0,\ 0,\ \frac{2\cdot Q_5}{\delta},\ -\frac{20\,x^3\cdot Q_6}{\delta^2},\ 0,\ 0, -\frac{2\cdot Q_5}{\delta},\ \frac{12\,x^3\,Q_6}{\delta^2},\ -\frac{2\,x^6\,Q_{11}}{\delta^3},\ \frac{Q_5}{\delta},$$
$$-\frac{4\,x^7\cdot Q_6}{\delta^2},\ \frac{2\,x^6\,Q_{14}}{\delta^3},\ \frac{6\,x^9\,Q_{15}}{\delta^4},\ \frac{x^6\,Q_{16}}{\delta^3},\ \frac{-2\,x^9\,Q_{17}}{\delta^4},\ \frac{8\,x^{12}\,Q_{18}}{\delta^5},\ \frac{x^{12}\,Q_{19}}{\delta^5},$$
$$\frac{-8\,x^{15}\,Q_{20}}{\delta^6},\ \frac{4\,x^{18}\,Q_{21}}{\delta^6}\Big],$$

where

$$c_1 = c_2 = c_3 = c_4 = c_7 = c_8 = 0, \qquad c_5 = c_9 = c_{12}, \tag{64}$$

and where, recalling p_2 in (51) and λ in (48)

$$\delta = -x^4\cdot\lambda(x), \qquad Q_5 = p_2,$$

$$Q_6 = 14495514624\,x^8 - 8191475712\,x^7 + 1552941056\,x^6 - 94273536\,x^5$$
$$- 3440640\,x^4 + 498624\,x^3 - 3632\,x^2 - 609\,x + 9,$$

¶ In fact, an argument of Katz [7] enables, in principle, to see whether the differential Galois group is included in $O(6, \mathbb{C})$ or actually equal to $O(6, \mathbb{C})$. This argument is difficult to work out here.

‡ See Katz's book [7] and most of the explicit examples known in the literature.

§ In order to do these calculations on the linear differential systems, download the Maple Tools files TensorConstructions.m and IntegrableConnections.m in the web page [41]. Using DEtools, you will need to use, on the order-six operator G_6^{5Dfcc}, the command companion-system,then the command symmetric-power-system(,2)) and finally the command RationalSolutions([],[x]).

the other Q_n's being much larger polynomials.

If one wants to stick with an operator description, similarly to (35) or (44), one can switch, by operator equivalence†, to an operator such that its symmetric square is of the generic order 21 and has a *rational* solution.

The denominator of the monic order-twenty operator $Sym^2(G_6^{5Dfcc})$ is of the form $x^{16} \cdot \lambda(x)^5 \cdot p_{278}$, where p_{278} is a polynomial of degree 278 in x.

Let us introduce, for $n \ge 2$, an equivalent operator $G_6^{(n)}$, corresponding to an intertwining by D_x^n

$$S_2^{(n)} \cdot G_6^{5Dfcc} \quad = \quad G_6^{(n)} \cdot D_x^n, \qquad (65)$$

For $n = 2$, the *symmetric square* of $G_6^{(n)}$ has the *rational solution*

$$\frac{p_2}{x^4 \cdot \lambda(x)} \, = \, \frac{1152\, x^2 - 56\, x - 3}{x^4 \cdot \, (16\, x + 1)\, (8\, x - 1)\, (4\, x - 1)\, (16\, x - 1)\, (48\, x - 1)\, (16\, x - 3)}, \qquad (66)$$

which is nothing but $c_5/2$ in (63). For the symmetric square of the other $\tilde{G}_6^{(n)}$'s one finds, respectively for $n = 3$, $n = 4$ and $n = 5$, the rational solutions c_{16}, c_{19} and c_{21} in (63). More generally the rational solution reads:

$$\frac{P_{12\,n-22}(x) \cdot \, x^{8\,n-12}}{x^{2\,n} \cdot \, \delta^{2\,n-3}}, \qquad (67)$$

where $P_m(x)$ is a polynomial of degree m in x.

Getting (or even only checking) the rational solution (66) for the symmetric square of the equivalent operator (65), paradoxically, corresponds to massive calculations compared to obtaining the rational solution on the symmetric square of the companion system (see (63)).

3.9. Koutschan's order-eight operator: the lattice Green function of the six-dimensional fcc lattice

A slightly more spectacular‡ example of order-eight, G_8^{6Dfcc}, has been found by Koutschan [27] for a *six-dimensional face-centered cubic lattice.* Its *exterior square is of order 27.* The irreducibility of this order-eight operator is hard to check§. One finds again, at the origin $x = 0$, that there are two independent analytical solutions (no logarithms). Since the order-eight operator G_8^{6Dfcc} has *two analytical solutions*, it *cannot be* MUM [19] at $x = 0$.

A linear combination of these solutions is globally bounded [18, 19]. It is such that, after rescaling, it can be recast into a series with *integer* coefficients [18, 19]:

$$1 \, + 60\, x^2 \, + 960\, x^3 \, + 30780\, x^4 + 996480\, x^5 \, + 36560400\, x^6 \, + 1430553600\, x^7$$
$$+ \, 59089923900\, x^8 \, + 2543035488000\, x^9 \, + 113129280527760\, x^{10} \, + \, \cdots \qquad (68)$$

In order to fix the normalization, this order-eight operator will be analyzed in a monic form: $G_8^{6Dfcc} \, = \, D_x^8 \, + \, \cdots$ This order-eight operator is (non-trivially)

† In Maple, choosing an intertwiner R, e.g. $R = D_x^2$, this amounts to performing $S_2 =$ rightdivision(LCLM(G_6^{5Dfcc}, R), R). Note that the LCLM with D_x instead of D_x^2, still yields a symmetric square of order 20 instead of 21.

‡ It is a quite large [28] order-eight linear differential operator of 52 Megabytes.

§ One can, however, check that this operator has no rational solutions.

homomorphic to its adjoint, one intertwiner being of order *six*, the other one being of order *two*

$$adjoint(S_6^{6Dfcc}) \cdot G_8^{6Dfcc} \;=\; adjoint(G_8^{6Dfcc}) \cdot S_6^{6Dfcc}, \qquad (69)$$

$$G_8^{6Dfcc} \cdot T_2^{6Dfcc} \;=\; adjoint(T_2^{6Dfcc}) \cdot adjoint(G_8^{6Dfcc}). \qquad (70)$$

where, again, noticeably, the two intertwiners are, after the *same* rescaling *self-adjoint operators*. Let us introduce

$$a(x) \;=\; \frac{x^6 \cdot p_5^4}{p_{25}} \cdot \lambda(x), \qquad (71)$$

where the polynomial p_5 reads

$$p_5 \;=\; 56\,x^5 \;+625\,x^4 \;-1251\,x^3 \;-24840\,x^2 \;-65556\,x \;-38880, \qquad (72)$$

where $\lambda(x)$ reads:

$$\lambda(x) \;=\; (x-1)\,(x-3)\,(x+24)\,(2\,x\,+15)\,(7\,x+60)\,(2\,x\,+3)\,(4\,x\,+15) \times\; (x\,+9)\,(x\,+5)\,(x\,+4)\,(x\,+15)^4, \qquad (73)$$

and where the polynomial p_{25} is a quite large polynomial of degree 25.

The intertwiners T_2^{6Dfcc} and S_6^{6Dfcc} are, respectively, of the form $T_2^{6Dfcc} = a(x) \cdot Y_2^s$ and $S_6^{6Dfcc} = a(x) \cdot Y_6^s$, where Y_2^s and Y_6^s are two irreducible *self-adjoint* order-two and order-six operators

$$Y_2^s \;=\; \frac{1}{W_2(x)} \cdot \Big(D_x^2 \;-\; \frac{d\ln(W_2(x))}{dx} \cdot D_x \;+\; \cdots\Big), \qquad (74)$$

and:

$$Y_6^s \;=\; \frac{1}{W_6(x)^{1/3}} \cdot \Big(D_x^6 \;-\; \frac{d\ln(W_6(x))}{dx} \cdot D_x^5 \;+\; \cdots\Big). \qquad (75)$$

Their corresponding two Wronskians $W_2(x)$ and $W_6(x)$ read respectively:

$$W_2(x) \;=\; \frac{x^{11} \cdot \lambda(x)^2 \cdot p_5^3}{(x+15)^3 \cdot p_{25}}, \qquad W_6(x)^{1/3} \;=\; \frac{(x+15)^3 \cdot p_5}{\lambda(x) \cdot x^5}. \qquad (76)$$

These self-adjoint operators are *not globally nilpotent* [8], however we see, again, the emergence of some structure: the (Jordan form of the) p-curvature of these two self-adjoint operators read

$$\left[\begin{array}{cc} 0 & 1 \\ S_1^p & 0 \end{array}\right], \qquad \left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ R_1^p & 0 & R_2^p & 0 & R_3^p & 0 \end{array}\right] \qquad (77)$$

where S_1 is a different rational function for each prime p, for instance for $p \,=\, 11$

$$S_1 \;=\; \frac{4 \cdot (5+x)^2}{(x+7) \cdot x \cdot (x^2+7\,x+5) \cdot (x+4) \cdot (x+2)}, \qquad (78)$$

and similarly for the R_n's of the 6×6 matrix in (77).

The intertwining relations (69) give, in terms of the self-adjoint operators (74) and (75):

$$Y_6^s \cdot a(x) \cdot G_8^{6Dfcc} \;=\; adjoint(G_8^{6Dfcc}) \cdot a(x) \cdot Y_6^s,$$
$$G_8^{6Dfcc} \cdot a(x) \cdot Y_2^s \;=\; Y_2^s \cdot a(x) \cdot adjoint(G_8^{6Dfcc}), \qquad (79)$$

which yield

$$\mathcal{K}_8 \cdot \mathcal{M}_8 \;=\; \mathcal{M}_8 \cdot \mathcal{K}_8 \qquad \hbox{and} \qquad (80)$$
$$adjoint(\mathcal{M}_8) \cdot adjoint(\mathcal{K}_8) \;=\; adjoint(\mathcal{K}_8) \cdot adjoint(\mathcal{M}_8), \qquad \hbox{where:}$$
$$\mathcal{K}_8 \;=\; a(x) \cdot G_8^{6Dfcc} \qquad \hbox{and:} \qquad \mathcal{M}_8 \;=\; a(x) \cdot Y_2^s \cdot a(x) \cdot Y_6^s. \qquad (81)$$

A commutation relation of linear differential operators, like (80), is a drastic constraint on the operators. As $\mathcal{K}_8$ is *irreducible*, the commutation (80) forces $\mathcal{K}_8$ to be of the form $\alpha \cdot \mathcal{M}_8 + \beta$, where α and β are constants. We may thus guess, from the intertwining relations (79), a decomposition of the order-eight operator G_8^{6Dfcc}, similar to the one we had for G_6^{5Dfcc}, of the form

$$G_8^{6Dfcc} \;=\; Y_2^s \cdot a(x) \cdot Y_6^s \;+\; \frac{\alpha}{a(x)}, \qquad (82)$$

where Y_2^s and Y_6^s are two self-adjoint operators of *even* order (instead of odd order for G_6^{5Dfcc}). This is, indeed, the case. The operator G_8^{6Dfcc} has the noticeable decomposition:

$$G_8^{6Dfcc} \;=\; Y_2^s \cdot a(x) \cdot Y_6^s \;+\; \frac{87480}{a(x)}. \qquad (83)$$

Again, and similarly to what has been done for G_6^{5Dfcc} (see (61)), one can rewrite (83) as:

$$a(x) \cdot Y_2^s \cdot a(x) \cdot Y_6^s \;=\; -87480 \;+ a(x) \cdot G_8^{6Dfcc}, \qquad (84)$$

which means that the two intertwiners $a(x) \cdot Y_2^s$ and $a(x) \cdot Y_6^s$ are inverse of each other modulo the operator $a(x) \cdot G_8^{6Dfcc}$. From (84) one sees that a solution of G_8^{6Dfcc} is an eigenfunction of $a(x) \cdot Y_2^s \cdot a(x) \cdot Y_6^s$ with the eigenvalue -87480.

The examination of the formal series solutions, at $x = 0$, of the self-adjoint order-six Y_6^s operator shows a MUM structure. The Y_6^s operator has one analytic solution at $x = 0$, (the other solutions have log terms), which has the following expansion:

$$Sol(Y_6^s) \;=\; 1 \;+ \frac{197}{11520}\, x^2 \;+ \frac{8559443}{1889568000}\, x^3 \;+ \frac{381585241573}{154793410560000}\, x^4$$
$$+ \frac{35207145815207429}{27209779200000000000}\, x^5 \;+ \frac{15094430772106074099089}{18280792200314880000000000}\, x^6 \;+\; \cdots \qquad (85)$$

This solution-series (85), again, is *not*† *globally bounded* [18, 19]. One deduces immediately, from decomposition (83), an interesting eigenvalue result: the order-eight operator $a(x) \cdot G_8^{6Dfcc}$ has the *not globally bounded* eigenfunction (85), corresponding to the *integer eigenvalue* 87480.

The self-adjoint order-five irreducible operator Y_6^s is such that its *exterior square is of order 14 instead of the order 15 expected generically* (its symmetric square is of order 21 as it should, with no rational solution).

† This is also the case for the self-adjoint order-two Y_2^s operator. Its solution analytic at $x = 0$ is *not* globally bounded [18, 19].

The symmetric square of G_8^{6Dfcc} is of the (generic) order 36. However the *exterior square* of G_8^{6Dfcc} is of order 27 *instead of the (generic) order* 28.

Remark: The adjoint of G_8^{6Dfcc} has the following decomposition, straightforwardly deduced from (83):

$$adjoint(G_8^{6Dfcc}) \quad = \quad Y_6^s \cdot a(x) \cdot Y_2^s \quad + \quad \frac{87480}{a(x)}. \tag{86}$$

So we can expect the Wronskian of Y_2^s to be a (rational) solution of its exterior square. We have verified that *this is indeed the case.*

Similarly to the previous order-six operator G_6^{5Dfcc}, one could try to switch to equivalent operators (see (65)), calculate the exterior square of these equivalent operators, and try to find the corresponding rational solution (see (66)). These, at first sight, straightforward calculations are, in fact, too "massive". The way to get the rational solution is, in fact, to switch to *differential systems* (see (63)).

3.9.1. System representation of G_8^{6Dfcc}

In fact, even after switching to a differential system using the same tools [41] that we used for obtaining (63), we found that the resulting calculation exceeded our computational capacity. These calculations mostly amount to finding a transformation that reduces the system to a system with simple poles. We need a second "trick" to be able to achieve these calculations and get the rational solution of the differential system. An easy way is to rewrite the system‡ in terms of the *homogeneous derivative* $\theta \, = \, x \cdot D_x$. Switching to this companion system§ in θ, one automatically has simple poles for the system.

With all these tricks and tools, we finally found that the linear *differential system* for the *exterior square* of the order-eight G_8^{6Dfcc} operator is of *order 28 and has a rational solution.* Note that the intertwiners (69) have been found from this rational solution: seeking straightforwardly for the intertwiners (69) needs too massive calculations !

The rational solution reads

$$[c_1, \quad c_2, \quad c_3, \quad \cdots, \quad c_{28}] \quad = \tag{87}$$

$$= \quad \Big[0,\ 0,\ 0,\ 0,\ \frac{P_5}{\delta},\ \frac{x^4\,P_6}{\delta^2},\ \frac{x^8\,P_7}{\delta^3},\ 0,\ 0,\ \frac{P_{10}}{\delta},\ \frac{x^4\,P_{11}}{\delta^2},\ \frac{x^8\,P_{12}}{\delta^3},\ \frac{x^{12}\,P_{13}}{\delta^4},$$
$$\frac{P_{14}}{\delta},\ \frac{x^4\,P_{15}}{\delta^2},\ \frac{x^8\,P_{16}}{\delta^3},\ \frac{x^{12}\,P_{17}}{\delta^4},\ \frac{x^{16}\,P_{18}}{\delta^5},\ \frac{x^3\,P_{19}}{\delta^2},\ \frac{x^7\,P_{20}}{\delta^3},\ \frac{x^{11}\,P_{21}}{\delta^4},$$
$$\frac{x^{15}\,P_{22}}{\delta^5},\ \frac{x^6\,P_{23}}{\delta^3},\frac{x^{10}\,P_{24}}{\delta^4},\ \frac{x^{14}\,P_{25}}{\delta^5},\ \frac{x^9\,P_{26}}{\delta^4},\ \frac{x^{13}\,P_{27}}{\delta^5},\ \frac{x^{12}\,P_{28}}{\delta^5}\Big],$$

where

$$\delta \quad = \quad (x-1)\,(x-3)\,(x+24)\,(2\,x\,+15)\,(7\,x\,+60)\,(2\,x\,+3)$$
$$\times\,(x\,+15)\,(4\,x\,+15)\,(x\,+9)\,(x\,+5)\,(x+4)\cdot x^5, \tag{88}$$

‡ In the Maple TensorConstruction tools found at [41], the command `Theta_companion_system`(L) returns two matrices $\frac{1}{p(x)} A_\theta$ and P_θ such that, for $Y \, = \, (y, y', \ldots, y^{(n-1)})^T$, we have $Y \, = \, P_\theta Y_\theta$ and $Y'_\theta \, = \, \frac{1}{p(x)} \, A_\theta \, Y_\theta$, where A_θ has no finite poles and $p(x)$ is squarefree, it has only simple roots. This gives the correspondence between the original companion system and the θ-companion system.
§ If one is reluctant to switch to companion systems in θ, another way to achieve these calculations is to perform a reduction on the matrix of the corresponding system (moser-reduce in Maple) *before* calculating the symmetric powers of the system (an operation that preserves the order of the poles).

and where the polynomial P_n in (87) are too large to be displayed here. The polynomials $P_{28}, P_{27}, P_{25}, P_{22}, P_{18}$ are of degree 49, polynomials $P_{26}, P_{24}, P_{21}, P_{17}$ are of degree 38, P_{13} is of degree 37, polynomials $P_{23}, P_{20}, P_{16}, P_{12}, P_7$ are of degree 27, polynomials $P_{19}, P_{15}, P_{11}, P_6$ are of degree 16, and P_{14}, P_{10}, P_5 are of degree 5. Furthermore we have some equalities like $P_{14} = -P_{10} = P_5$, $P_{11} = -2 \cdot P_{15} = -2/3 \cdot P_6$

Having this rational solution at our disposal, we can, *now*, find the rational solutions for the exterior square of the equivalent operators:

$$G_2^{(n)} \cdot G_8^{6Dfcc} \quad = \quad G_8^{(n)} \cdot D_x^n. \tag{89}$$

Recalling (72), the rational solution for $n = 2$ reads p_5/δ. The differential Galois group of G_8^{6Dfcc} is included in (and probably equal to) $Sp(8, C)$.

Remark: The same calculations can be performed on all the linear differential operators we have encountered in lattice statistical mechanics [8, 11, 12, 13, 14, 15, 16, 17, 19, 20]: all the examples we have tested give operators whose irreducible factors are actually equivalent to their adjoint.

3.10. Generalization of the decomposition for higher order operators

The remarkable decompositions (60) and (83), encountered with G_6^{5Dfcc} and G_8^{6Dfcc} can easily be generalized. In fact, one can systematically introduce the *even* order operators

$$M_{2p}^{(n,\,2p-n)} \quad = \quad L_{2p-n} \cdot a(x) \cdot L_n \; + \; \frac{\lambda}{a(x)}, \tag{90}$$

or, after rescaling‡,

$$\tilde{M}_{2p}^{(n,\,2p-n)} \quad = \quad a(x) \cdot L_{2p-n} \cdot a(x) \cdot L_n \; + \; \lambda, \tag{91}$$

where the L_m's are *self-adjoint operators* of order m. They are, naturally, homomorphic to their adjoint, with intertwiners corresponding to these decompositions (90) and (91):

$$a(x) \cdot L_n \cdot M_{2p}^{(n,\,2p-n)} \quad = \quad adjoint(M_{2p}^{(n,\,2p-n)}) \cdot a(x) \cdot L_n,$$
$$M_{2p}^{(n,\,2p-n)} \cdot a(x) \cdot L_{2p-n} \quad = \quad L_{2p-n} \cdot a(x) \cdot adjoint(M_{2p}^{(n,\,2p-n)}),$$

$$L_n \cdot \tilde{M}_{2p}^{(n,\,2p-n)} \quad = \quad adjoint(\tilde{M}_{2p}^{(n,\,2p-n)}) \cdot L_n,$$
$$\tilde{M}_{2p}^{(n,\,2p-n)} \cdot a(x) \cdot L_{2p-n} \cdot a(x) \quad = \quad a(x) \cdot L_{2p-n} \cdot a(x) \cdot adjoint(\tilde{M}_{2p}^{(n,\,2p-n)}).$$

Experimentally we have seen (for instance with our two previous lattice Green functions examples of order six and eight, see (57) and (58) for (60), and (69) and (70) for (83)), that this corresponds to *two different types* of operators: the operators with *even* n, for which the *exterior square* of an equivalent operator (or of the corresponding differential system) will have a rational solution (yielding a symplectic differential Galois group), and the operators with *odd* n, for which the *symmetric square* of the corresponding differential system will have a rational solution (yielding an orthogonal differential Galois group).

‡ Do note that the rescaled operators (91) are such that the functions annihilated by L_n are automatically *eigenfunctions* of $\tilde{M}_{2p}^{(n,\,2p-n)}$ with *eigenvalue* λ.

4. Focus on order-four differential operators: Calabi-Yau conditions

It has been underlined by A.J. Guttmann that these lattice Green functions are (most of the time) solutions of Calabi-Yau ODEs, or higher order Calabi-Yau ODEs [2, 3]. The definition of Calabi-Yau ODEs, and some large lists of Calabi-Yau ODEs, can be found in [4, 5, 43, 44]. Calabi-Yau ODEs are defined by several constraints, some are natural like being MUM, others (like some cyclotomic constraints) are essentially introduced, in a classification perspective like [5] to provide hopefully exhaustive lists of Calabi-Yau ODEs, some are related to the concept of "*modularity*", requiring the *integrality* of various series like the nome or the Yukawa coupling. Therefore, in the definition of Calabi-Yau ODEs, there is some "mix" between analytic and differential constraints, and constraints of a more arithmetic†, or algebraic geometry character. Among all these constraints defining the Calabi-Yau ODEs, the most important one is the so-called "*Calabi-Yau condition*". Let us consider a (monic) order-four linear differential operator:

$$\Omega_4 \quad = \quad D_x^4 \; + \; a_3(x) \cdot D_x^3 \; + \; a_2(x) \cdot D_x^2 \; + \; a_1(x) \cdot D_x \; + \; a_0(x). \qquad (92)$$

The exterior square of (92), $Ext^2(\Omega_4)$, reads, up to an overall factor:

$$C_6(x) \cdot Ext^2(\Omega_4) \quad = \quad \sum_{n=0}^{6} C_n(x) \cdot D_x^n, \qquad (93)$$

where the $C_i(x)$'s are polynomial expressions of $a_3(x)$, $a_2(x)$, $a_1(x)$, $a_0(x)$ and of their derivatives (up to the third derivative).

The vanishing condition $C_6(x) \, = \, 0$, which reads

$$8\, a_1(x) \; + \; a_3(x)^3 \; - 4 \cdot a_3(x) \cdot a_2(x) \; + 6 \cdot a_3(x) \cdot {{ d a_3(x)} \over {dx}} \\ - 8 \cdot {{d a_2(x)} \over {dx}} \; + 4 \cdot {{d^2 a_3(x)} \over {dx^2}} \quad = \quad 0, \qquad (94)$$

is satisfied if, and only if, the exterior square is of *order five*, instead of the order six one expects generically. It is called "*Calabi-Yau condition*" by some authors [6] and is one of the conditions for the ODE to be a *Picard-Fuchs equation* of a family of Calabi-Yau manifolds (see (11) in [45]). This Calabi-Yau condition (94) is actually *preserved by pullbacks, but not by operator equivalence*. Note that this Calabi-Yau condition (94) is *preserved by the adjoint transformation* (see Appendix B.3).

Do note that such condition is actually independent of $a_0(x)$ in (92). Also note that all the order-four operators M_4 that can be written as the sum of the symmetric-cube of an order-two operator¶, and a function, $M_4 \, = \, Sym^3(M_2) + \, f(x)$, automatically verify the Calabi-Yau condition (94). This gives a practical way to quickly provide examples of order-four operators satisfying the Calabi-Yau condition (94).

Of course similar Calabi-Yau conditions can be introduced for higher order operators, imposing, for order-N operators, that their *exterior squares* are of order *less than* the generic $N \cdot (N-1)/2$ order. These higher order *Calabi-Yau conditions* actually correspond to consider *self-adjoint operators* (see Appendix B.1, see also [22]).

† In order to disentangle these various constraints see [18, 19].

¶ If the order-two operator M_2 is chosen to be globally nilpotent its Wronskian is an N-th root of a rational function, and $M_4 \, = \, Sym^3(M_2) \, + \, f(x)$ is conjugated to its adjoint up to the cube of this Wronskian.

Furthermore similar "Calabi-Yau conditions" can be introduced for *symmetric squares instead of exterior squares*, imposing, for order-N operators, that their symmetric squares are of order *less than* the generic $N \cdot (N+1)/2$ order. For an order-three operator written in a monic form

$$\Omega_3 \quad = \quad D_x^3 \; + \; a_2(x) \cdot D_x^2 \; + \; a_1(x) \cdot D_x \; + \; a_0(x), \qquad (95)$$

the "symmetric Calabi-Yau condition" reads‡:

$$4\, a_2(x)^3 \; - 18\, a_1(x)\, a_2(x) \; + 9 \cdot {{d^2 a_2(x)} \over {dx^2}} \; + 18 \cdot a_2(x) \cdot {{d a_2(x)} \over {dx}} \\ + 54\, a_0(x) \; - 27\, {{d a_1(x)} \over {dx}} \quad = \quad 0. \qquad (96)$$

Operators satisfying this "symmetric Calabi-Yau condition" actually correspond to the situation described in (3.4). If their Wronskian $W(\Omega_3)$ is a N-th root of a rational function they are conjugated to their adjoint ($f(x) \, = \, W(\Omega_3)^{2/3}$):

$$\Omega_3 \cdot f(x) \quad = \quad f(x) \cdot adjoint(\Omega_3) \qquad \text{with:} \qquad a_2(x) \, = \, - {{3} \over {2}} {{1} \over {f(x)}} {{df(x)} \over {dx}}. \qquad (97)$$

In order to disentangle the main focus of this very paper, namely the *algebraic-differential structures*, from other structures of more analytical, or arithmetic, character (series integrality [18, 19], MUM property, etc.), we concentrate, in this section, on (mostly order-four) linear differential operators satisfying the Calabi-Yau condition (94), or *homomorphic to operators satisfying* (94).

4.1. Weak and strong Calabi-Yau conditions

If one considers an operator that is homomorphic to an operator with a *rational Wronskian*, satisfying the Calabi-Yau condition (94), with intertwiners that are of order greater or equal to one†, the exterior square of that operator actually *has a rational solution.* Unfortunately, in contrast with (94), the condition for an order-four operator to be such that its exterior square has a rational solution, *cannot be written directly and explicitly on its coefficients* $a_n(x)$ (see (92)). We will call "weak Calabi-Yau condition" this condition that the exterior square of an operator has a rational solution, the Calabi-Yau condition (94) being the "strong" Calabi-Yau condition. Note that the weak Calabi-Yau condition is *preserved by the adjoint transformation* (see Appendix B.3).

As far as classifications of Calabi-Yau operators are concerned [4, 5, 43, 44], an operator non-trivially§ homomorphic to a "Calabi-Yau operator" is certainly as interesting for physics as these "Calabi-Yau operators", and an operator non-trivially homomorphic to an operator verifying the "strong" Calabi-Yau condition (94), or satisfying the "weak Calabi-Yau condition" is certainly as interesting as an operator verifying the "strong" Calabi-Yau condition.

Let us explore the relation between the "weak Calabi-Yau condition" and the "strong Calabi-Yau condition".

‡ The "symmetric Calabi-Yau condition" for order-four operators can be found but is drastically larger than (96).

† We must exclude intertwiners of order zero (namely functions): in that case, it is a straightforward calculation to see that the operators are conjugated by a function, both operators satisfying the Calabi-Yau condition (94).

§ With intertwiners of order greater or equal to one.

4.2. A decomposition of operators equivalent to operators satisfying the Calabi-Yau condition

Let us consider an order-four operator Ω_4, of Wronskian $w(x) \;=\; u(x)^2$, which satisfies the Calabi-Yau condition (94). Let us also consider a monic order-four operator $\tilde{\Omega}_4$ which is (non-trivially) homomorphic (equivalent in the sense of the equivalence of operators [31]) to the order-four operator Ω_4 satisfying the Calabi-Yau condition (94). This amounts to saying that there exist two intertwiners, U_3 and L_3, of order less or equal to three‡, such that:

$$\tilde{\Omega}_4 \cdot U_3 \quad = \quad L_3 \cdot \Omega_4 \tag{98}$$

It is shown in Appendix A that the order-four operator $\tilde{\Omega}_4$ *can, in fact, be written in terms of a remarkable decomposition*

$$\tilde{\Omega}_4 \quad = \quad Z_2^s \cdot \frac{1}{A_0} \cdot A_2 \quad + \quad A_0, \tag{99}$$

where Z_2^s and A_2 are two *self-adjoint operators*, A_0 being a function. Appendix A shows how to get Z_2^s, A_2 and A_0 in such a decomposition: they can simply be obtained as the intertwiners of $\tilde{\Omega}_4$ with its adjoints (use (A.10), (A.13), (A.15), (A.16) in Appendix A). Experimentally we have checked that an operator (non trivially) homomorphic to an operator of the form (99) (see (A.2)) can always be decomposed in a form (99): *the decomposition (99) is preserved by operator equivalence.*

Byproduct: As a byproduct one finds out that the left and right intertwiners of an order-four operator satisfying the weak Calabi-Yau condition are *necessarily of order two.* Note, however, that this is not true for the intertwiners of an order-four operator satisfying the symmetric weak Calabi-Yau condition which are of odd orders (see (117) in the section 4.5 on the anisotropic simple cubic lattice Green function).

4.3. Rational solutions for the exterior square of operators satisfying the weak Calabi-Yau condition

We have the following general result. Any order-four linear differential operator of the form†

$$M_4 \quad = \quad L_2 \cdot c_0(x) \cdot M_2 \quad + \quad \frac{\lambda}{c_0(x)}, \tag{100}$$

where L_2 and M_2 are two (general) self-adjoint operators

$$L_2 \quad = \quad \alpha_2(x) \cdot D_x^2 \; + \; \frac{d\,\alpha_2(x)}{dx} \cdot D_x \; + \; \alpha_0(x), \tag{101}$$

$$M_2 \quad = \quad \beta_2(x) \cdot D_x^2 \; + \; \frac{d\,\beta_2(x)}{dx} \cdot D_x \; + \; \beta_0(x), \tag{102}$$

is such that *its exterior square has* $1/\beta_2(x)$ *as a solution* (up to an overall multiplicative constant). This result can be seen to be the consequence of a non trivial identity (B.25) given in Appendix B.4.

Byproduct: Thus the *exterior square* of $\tilde{\Omega}_4$ has a *rational solution*, which is the inverse of the head polynomial of the second order self-adjoint operator A_2 in the decomposition (99).

‡ Higher order intertwiners can always be reduced to intertwiners with an order less, or equal, to three.

† Note that one can always restrict to $\lambda \,=\, 1$ rescaling $c_0(x)$ into $\lambda^{1/2} \cdot c_0(x)$.

To sum-up: The operators, non-trivially homomorphic to operators satisfying the (strong) Calabi-Yau condition (94), necessarily satisfy the "weak Calabi-Yau condition": their exterior square have a *rational solution*. Furthermore this rational solution corresponds to the Wronskian of a self-adjoint order-two operator L_2 of a remarkable decomposition (100). Decomposition (100) is the most general form of an operator satisfying the "weak Calabi-Yau condition".

Conversely: This naturally raises the reciprocal question. Is any order-four operator satisfying the "weak Calabi-Yau condition" (its exterior square has a rational solution) non trivially homomorphic to an operator satisfying the (strong) Calabi-Yau condition (94) ? In view of the remarkable decomposition (100), we can also ask the following questions. Is any order-four operator satisfying the "weak Calabi-Yau condition" necessarily of the form (100), i.e. homomorphic to its adjoint with *order-two* intertwiners ? Is any order-four operator of the form (100) homomorphic to an operator satisfying the (strong) Calabi-Yau condition (94) ? These questions will be revisited in a forthcoming publication‡. The reason why these questions are difficult to answer in general, beyond specific examples, comes from the fact that such a reduction by operator equivalence of operators satisfying the weak Calabi-Yau condition to operators satisfying the strong Calabi-Yau condition, is *not unique* (an infinite number of equivalent operators can satisfy the Calabi-Yau condition (94), see Appendix B.2).

4.4. Calabi-Yau conditions and rational solutions of the exterior square for order-one intertwiners

Let us consider an order-four operator Ω_4, satisfying the Calabi-Yau condition (94), and let us introduce $u(x)$ the square root of its Wronskian: $w(x) = u(x)^2$. Let us consider the LCLM of Ω_4 and of the order-one operator D_x. This LCLM reads:

$$L_1 \cdot \Omega_4 \quad = \quad M_4 \cdot D_x \qquad \text{where:} \qquad L_1 \; = \; D_x \; - \; \frac{1}{a_0(x)} \cdot \frac{da_0(x)}{dx}. \tag{103}$$

The order-four operator M_4 verifies the "weak Calabi-Yau condition". Its exterior square has $u(x)$ as a solution. More precisely, the exterior square of M_4 is, in fact, the *direct sum* of an order-one operator and of an order-five operator

$$Ext^2(M_4) \; = \; M_5 \oplus M_1 \qquad \text{where:} \qquad M_1 \; = \; D_x \; - \; \frac{1}{u(x)} \cdot \frac{du(x)}{dx}, \tag{104}$$

where the order-five operator is homomorphic to the exterior square of Ω_4, with two order-two intertwiners U_2 and V_2, namely $M_5 \cdot U_2 = V_2 \cdot Ext^2(\Omega_4)$, the order-two intertwiner U_2 reading

$$U_2 \; = \; D_x^2 \; - \; \frac{1}{u(x)} \cdot \frac{du(x)}{dx} \cdot D_x \; + r(x) \qquad \text{where:}$$

$$r(x) \; = \; a_2(x) \; + \frac{1}{u(x)} \cdot \frac{d^2u(x)}{dx^2} \; - 2 \cdot \Big(\frac{1}{u(x)} \cdot \frac{du(x)}{dx}\Big)^2. \tag{105}$$

‡ If one switches to a representation in terms of *differential systems*, such a system with Galois group $Sp(4, \mathbb{C})$ can always be reduced, via a "gauge-like" transformation [46, 47], to a system with a *hamiltonian* matrix M. Such a system is such that the exterior square system associated with a 6×6 matrix, has a constant solution, namely [0, 1, 0, 0, 1, 0] (see [46, 47]). Switching back to the operator representation, one actually finds that this operator is homomorphic to its adjoint with order-two intertwiners (themselves homomorphic to their adjoints). Consequently they can always be decomposed into a form (90).

These results can be generalized to more general order-one intertwiners. One can easily deduce the result (here O^F denotes the conjugate by $F(x)$ of an operator O: $O^F \, = \, F(x) \cdot O \cdot F(x)^{-1}$)

$$L_1^F \cdot \Omega_4^F \quad = \quad M_4^F \cdot \Big(D_x \, - \, {{1} \over {F(x)}} \cdot {{dF(x)} \over {dx}} \Big), \qquad (106)$$

where, again, Ω_4^F satisfies the Calabi-Yau condition (94), and where M_4^F satisfies the weak Calabi-Yau condition. Along this more general order-one line, see (B.24) in (Appendix B.3) and (108) below.

Switching to a linear differential system representation of an order-four operator satisfying the Calabi-Yau condition (94), one has the following solution for the differential system :

$$\Big[0, \,\, 0, \,\, -u(x), \,\,\, u(x), \,\,\, {{du(x)} \over {dx}}, \,\,\, r(x) \cdot u(x) \Big]. \qquad (107)$$

In that heuristic case one remarks that $A_0 \, = \, a_0(x)$ in (92), and that U_2 in (105) is simply related to A_2 in the decomposition (99): $U_2 \, = \, u(x) \cdot A_2$.

If one assumes that the Wronskian of the order-four linear differential operator is a *square of a rational function*, one, thus, finds a *rational solution for the exterior square of the differential system* (resp. hyperexponential solution [42] for a Wronskian N-th root of a rational function).

If one prefers to stick with differential *operators* instead of differential *systems*, one can see the emergence of a *rational solution* for the exterior square of the order-four operator, exchanging the order-four operator satisfying the Calabi-Yau condition (94) for an equivalent operator (in the sense of the equivalence of operators [31], see next subsection). This equivalent operator *does not satisfy the Calabi-Yau condition* (94) since, as we noticed, the Calabi-Yau condition (94) is *preserved by pullback* but *not by operator equivalence*.

Remark 1: In this case of an operator, like M_4 in (103), satisfying the weak Calabi-Yau condition i.e. homomorphic to an operator Ω_4 satisfying the Calabi-Yau condition (94) with an *order-one* intertwiner, one can, for a given M_4, find Ω_4 from (104), (105). In this case (*order-one* intertwiner) the order-four operator Ω_4 is (up to overall factors) unique.

Remark 2: Recalling the general decomposition result (99), one actually finds that an order-one intertwiner situation $L_1 \cdot \Omega_4 \, = \, M_4 \cdot (D_x + q_0(x))$, corresponds to the *self-adjoint operators* Z_2^s and A_2 in (99) of the form (Z_2^s factors in order-one operators††):

$$Z_2^s \,\,\, = \,\,\, u(x) \cdot A_0 \cdot \Big(D_x \, + {{1} \over {u(x)}} \cdot {{du(x)} \over {dx}} \, + q_0(x) \Big) \cdot \Big(D_x \, + {{1} \over {A_0}} \cdot {{dA_0} \over {dx}} \, - q_0(x) \Big),$$

$$u(x) \cdot A_2 \,\,\, = \,\,\, D_x^2 \,\, - {{1} \over {u(x)}} \cdot {{du(x)} \over {dx}} \cdot D_x \,\, + r(x), \qquad (108)$$

where $r(x)$ reduces to (105) when $q_0(x) \, = \, 0$.

Remark 3: These results are specific of order-one intertwiners. One can consider order-two intertwiners introducing the LCLM of Ω_4 and of an order-two operator M_2, yielding the intertwining relation $L_2 \cdot \Omega_4 \, = \, M_4 \cdot M_2$. Again one still has a decomposition (99) with Z_2^s and A_2, two order-two self-adjoint operators, but where Z_2^s no longer factorizes.

†† Operators Z_2^s and A_2 of the form (108) are automatically self-adjoint.

4.5. The lattice Green function of the anisotropic simple cubic lattice

At this step it is important to recall the results of Delves and Joyce [48, 49] for the lattice Green function of the *anisotropic simple cubic lattice*, generalizing the results displayed in section (3.1). The lattice Green function of that anisotropic lattice is solution of an order-four operator (see (14) in [49]), depending on an anisotropy parameter α. This order-four operator reads in terms of the homogeneous derivative $\theta = x \cdot D_x$:

$$\begin{aligned} G_4^{asc} = \, & 24 \cdot \theta^3 \cdot (\theta-1) \, - 4 \cdot x \cdot \theta \cdot P_1(\theta) \, + 2 \cdot x^2 \cdot (2\theta+1) \cdot P_2(\theta) \\ & - A \cdot x^3 \cdot (2\theta+3)(2\theta+1) \cdot P_3(\theta) \\ & + 5 \cdot (A+4) \cdot A^3 \cdot x^4 \cdot (2\theta+5)(2\theta+3)(2\theta+1)(\theta+1), \end{aligned} \tag{109}$$

where $A = \alpha^2 - 4$ and

$$\begin{aligned} P_1(\theta) &= 6 \cdot (2\theta+1)(10\theta^2+10\theta+3) \, + A \cdot (28\theta^3+7\theta^2+16\theta+3), \\ P_2(\theta) &= 12 \cdot (4\theta+5)(2\theta+3)(4\theta+3) \, + 2A \cdot \big(172\theta^3 + 252\theta^2+234\theta+81\big) \\ & \qquad + 3A^2 \cdot (16\theta^3+21\theta^2+18\theta+6), \\ P_3(\theta) &= 40 \cdot (4\theta+3)(4\theta+1) + 12A \cdot (22\theta^2+29\theta+12) \, + A^2 \cdot (36\theta^2+57\theta+31). \end{aligned}$$

Operator (109) is *globally nilpotent* [8, 40] for *any rational value* of the parameter A, the Jordan reduction of its p-curvature [8, 40] reading

$$J_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{110}$$

Its characteristic polynomial $P(\lambda)$ is λ^4, and its minimal polynomial is λ^3 (mod. any prime p).

This order-four operator G_4^{asc} is *not MUM*. It has two solutions analytic at $x = 0$

$$\begin{aligned} & 1 \, + \frac{1}{2}\,\big(\alpha^2+2\big) \cdot x \, + \frac{3}{8}\,\big(\alpha^4+8\alpha^2+6\big) \cdot x^2 \, + \frac{5}{16}\,\big(\alpha^6+18\alpha^4+54\alpha^2+20\big) \cdot x^3 \, + \, \cdots, \\ & x \, + \frac{3}{8}\,\big(3\alpha^2+11\big) \cdot x^2 \, + \frac{5}{48}\,\big(11\alpha^4+119\alpha^2+146\big) \cdot x^3 \\ & \qquad + \frac{35}{768}\,\big(25\alpha^6+537\alpha^4+2049\alpha^2+1217\big) \cdot x^4 \, + \, \cdots \end{aligned} \tag{111}$$

together with a solution with a log and a solution with a $\log^2$. The first analytic solution is *globally bounded* [18, 19] for generic rational values of α, or, even, generic rational values of A: for $A = p/q$ the rescaling $x \to 4q \cdot x$ changes this series into a series with *integer coefficients*. The second analytic solution (111) is *not globally bounded* for generic rational values of α, but becomes globally bounded for $\alpha = \pm 1$: with a rescaling $x \to 4x$, the series becomes a series with *integer coefficients*.

The exterior square of G_4^{asc} (depending on the parameter α) is of order six with no rational (or hyperexponential [24]) solutions. The *symmetric square* of G_4^{asc} is of order *nine*, instead of the order ten one could expect. In other words G_4^{asc} *verifies the symmetric Calabi-Yau condition* for order-four operators (see (96) above for order-three symmetric condition).

If, as previously done, we introduce an order-four operator $\tilde{G}_4^{asc}$ equivalent to G_4^{asc}

$$S_1^{asc} \cdot G_4^{asc} \;\; = \;\; \tilde{G}_4^{asc} \cdot D_x, \qquad (112)$$

the *symmetric square* of that equivalent order-four operator has a *rational solution* $r(x)$:

$$r(x) \;\; = \;\; \frac{(\alpha^2 - 4) \cdot x \; + 3}{x^2 \cdot (1 - \alpha^2 \cdot x) \cdot (1 - (\alpha - 2)^2 \cdot x) \cdot (1 - (\alpha + 2)^2 \cdot x)}. \qquad (113)$$

The order-four operator (109) can be decomposed in terms of two self-adjoint operators of order one and three, $Y_1^{(s)}$ and $Y_3^{(s)}$, namely

$$G_4^{asc} \; = \; Y_1^{(s)} \cdot \rho(x) \cdot Y_3^{(s)} \; + \; \frac{8 \cdot (\alpha^2 - 1)^2}{\rho(x)}, \quad \rho(x) \; = \; \frac{((\alpha^2 - 4) \cdot x + 3)^4}{(5\,(\alpha^2 - 4) \cdot x - 3)^3}, \quad (114)$$

where:

$$\rho(x) \cdot Y_1^{(s)} \;\; = \;\; 2 \cdot ((\alpha^2 - 4) \cdot x + 3) \;\; (5\; (\alpha^2 - 4) \cdot x - 3) \cdot D_x$$
$$+ \; (\alpha^2 - 4)\,(5\,(\alpha^2 - 4) \cdot x + 69), \qquad (115)$$

$Y_3^{(s)}$ being slightly more involved. One more time, and similarly to what has been done for G_6^{5Dfcc} and G_8^{6Dfcc} (see (61) and (84)), one can rewrite (114) as

$$\rho(x) \cdot Y_1^{(s)} \cdot \rho(x) \cdot Y_3^{(s)} \;\; = \;\; -8 \cdot (\alpha^2 - 1)^2 \;\; + \rho(x) \cdot G_4^{asc}, \qquad (116)$$

which means that the two intertwiners $\rho(x) \cdot Y_1^{(s)}$ and $\rho(x) \cdot Y_3^{(s)}$ are *inverse of each other modulo the operator* $\rho(x) \cdot G_4^{asc}$.

The order-four operator (109) is homomorphic to its adjoint with the intertwining relations:

$$Y_3^{(s)} \cdot \rho(x) \cdot G_4^{asc} \;\; = \;\; adjoint(G_4^{asc}) \cdot \rho(x) \cdot Y_3^{(s)},$$
$$G_4^{asc} \cdot \rho(x) \cdot Y_1^{(s)} \;\; = \;\; Y_1^{(s)} \cdot \rho(x) \; \cdot adjoint(G_4^{asc}). \qquad (117)$$

If one denotes W the Wronskian of G_4^{asc} one has the relation $r(x)^{20} \;\; = \;\; W^8 \cdot \rho(x)^5$ $(5\; (\alpha^2 - 4) \cdot x - 3)^7$.

Recalling the example of the order-six lattice Green operator G_6^{5Dfcc}, one sees that the fact that it is the *symmetric square* (and not the exterior square) of that order-four operator which has a rational solution, is related to the *odd order* of the two intertwiners. This anisotropic example shows that all the differential algebra structures we display in this paper *can be generalized, mutatis mutandis, to problems with more than one variable* (see also [50]).

5. Exceptional differential Galois groups

Recently a set of Calabi-Yau type operators whose differential Galois group is $G_2(C)$, the *exceptional*‡ subgroup [51] of $SO(7)$, were explicitly given [29, 30]. These examples read (see page 18 section 4.3 of [29], θ denotes the homogeneous derivative $\theta \; = \; x \cdot D_x$):

$$E_1 \;\; = \;\; \theta^7 \; - 128 \cdot x \cdot (48\,\theta^4 \; + 96\,\theta^3 \; + 124\,\theta^2 \; + 76\,\theta \; + 21)\,(2\,\theta \; + 1)^3$$
$$+ \, 4194304 \cdot x^2 \cdot (\theta \; + 1) \cdot (12\,\theta^2 \; + 24\,\theta \; + 23) \cdot (2\,\theta \; + 1)^2 \cdot (2\,\theta \; + 3)^2$$
$$- \, 34359738368 \cdot x^3 \cdot (2\,\theta \; + 5)^2 \cdot (2\,\theta \; + 1)^2 \cdot (2\,\theta \; + 3)^3, \qquad (118)$$

‡ The compact form of G_2, *subgroup of* $SO(7)$, can be described as the automorphism group of the octonion algebra.

and:

$$E_2 \;=\; \theta^7 \,-128 \cdot x \cdot (8\,\theta^4 \,+16\,\theta^3 \,+20\,\theta^2 \,+12\,\theta \,+3)\,(2\,\theta \,+1)^3 \\ \quad +1048576\,x^2 \cdot (2\,\theta \,+1)^2 \cdot (2\,\theta \,+3)^2 \cdot (\theta \,+1)^3,$$

$$E_3 \;=\; \theta^7 \\ \quad -3^3 \cdot x \cdot (81\,\theta^4 \,+162\,\theta^3 \,+198\,\theta^2 \,+117\,\theta \,+28)\,(2\,\theta \,+1)\,(3\,\theta \,+1)\,(3\,\theta \,+2) \\ \quad +3^{12}\,x^2 \cdot (3\,\theta \,+5) \cdot (3\,\theta \,+1) \cdot (\theta \,+1) \cdot (3\,\theta \,+2)^2 \cdot (3\,\theta \,+4)^2, \qquad (119)$$

$$E_4 \;=\; \theta^7 \\ \quad -2^7 \cdot x \cdot (128\,\theta^4 \,+256\,\theta^3 \,+304\,\theta^2 \,+176\,\theta \,+39)\,(4\,\theta \,+1)\,(4\,\theta \,+3)\,(2\,\theta \,+1) \\ \quad +2^{26}\,x^2 \cdot (4\,\theta \,+7)\,(4\,\theta \,+5)\,(4\,\theta \,+3)\,(4\,\theta \,+1)\,(2\,\theta \,+1)\,(2\,\theta \,+3)\,(\theta \,+1),$$

$$E_5 \;=\; \theta^7 \\ -2^7\,3^3 \cdot x \cdot (648\,\theta^4 \,+1296\,\theta^3 \,+1476\,\theta^2 \,+828\,\theta \,+155)\,(6\,\theta \,+5)\,(6\,\theta \,+1)\,(2\,\theta \,+1) \\ \quad +2^{20}\,3^{12}\,x^2 \cdot (6\,\theta \,+11)\,(6\,\theta \,+7)\,(6\,\theta \,+5)\,(6\,\theta \,+1)\,(3\,\theta \,+5)\,(3\,\theta \,+1)\,(\theta \,+1).$$

Note that for the five E_n, their conjugate $x^{-1/2} \cdot E_n \cdot x^{1/2}$ are *self-adjoint operators*, and of course† $x^{-1} \cdot E_n$ are *also self-adjoint operators*. Also note that the homogeneous derivative $\theta \,=\, x \cdot D_x$ is just shifted by $1/2$ by this conjugation:

$$x^{-1/2} \cdot \theta \cdot x^{1/2} \;=\; \theta \,+\, \frac{1}{2}. \qquad (120)$$

Therefore one gets easily the expressions of these new self-adjoint operators by changing θ into $\theta \,+\, 1/2$ in the previous definitions.

The Wronskians w_n of these order-seven operators E_n read respectively:

$$w_1 \;=\; \Bigl(\frac{1}{p_1^w}\Bigr)^{21/2} \qquad \hbox{and:} \qquad w_n \;=\; \Bigl(\frac{1}{p_n^w}\Bigr)^{7} \qquad n \,=\, 2,\, 3,\, 4,\, 5, \qquad (121)$$

where the p_n^w's are the following polynomials:

$$p_1^w \;=\; (16384\,x \,-1) \cdot x^2, \quad p_2^w \;=\; (4096\,x \,-1) \cdot x^3, \quad p_3^w \;=\; (19683\,x - 1) \cdot x^3, \\ p_4^w \;=\; (262144\,x \,-1) \cdot x^3, \qquad p_5^w \;=\; (80621568\,x \,-1) \cdot x^3. \qquad (122)$$

The solution-series, analytic at $x \,=\, 0$, of these order-seven operators E_n are actually series with *integer coefficients*. These series are displayed in Appendix C. These order-seven operators are MUM and are *globally nilpotent* (see (C.2) in Appendix C). The corresponding nomes (called "special coordinates" in [29]) defined as $q^{(n)} \;=\; \exp(y_1^{(n)}/y_0^{(n)}) \;=\; x \cdot \exp(\tilde{y}_1^{(n)}/y_0^{(n)})$, as well as the various *Yukawa couplings* [18, 19] of these order-seven operators, correspond to series with *integer coefficients* (see Appendix C).

Note that, after performing the following rescalings $x \;\rightarrow\; x/4096,\, x/19683$, $x/262144$, $x/80621568$ on the E_n's for $n \;=\; 2, 3, 4, 5$, these four rescaled E_n's have now the same Wronskian: $(1-x)^{-7} \cdot x^{-21}$. The homogeneous derivative being invariant by these rescalings the previous rescalings just amount to modifying the coefficients in front of the x^n's in the previous definitions, for instance:

$$E_2 \quad\longrightarrow\quad \hat{E}_2 \;=\; 2^5 \cdot \theta^7 \,-\, x \cdot (8\,\theta^4 \,+16\,\theta^3 \,+20\,\theta^2 \,+12\,\theta \,+3)\,(2\,\theta \,+1)^3 \\ \quad +2\,x^2 \cdot (2\,\theta \,+1)^2 \cdot (2\,\theta \,+3)^2 \cdot (\theta \,+1)^3.$$

† If an operator Ω is self-adjoint, the operator $f(x) \cdot \Omega \cdot f(x)$ is also self-adjoint for any function $f(x)$.

After these rescalings these rescaled operators $\hat{E}_i$ for $i = 2, \cdots 5$, have, now, their singularities in $0, 1$ and ∞. They read:

$$\hat{E}_i \quad = \quad (x-1)^2 \cdot x^7 \cdot D_x^7 \; + 7 \cdot (4x - 3)(x-1) \cdot x^6 \cdot D_x^6 \; + \cdots \qquad (123)$$

while for $\hat{E}_1$ one has:

$$\hat{E}_1 \quad = \quad (x-1)^3 \cdot x^7 \cdot D_x^7 \; + \frac{21}{2}(3x - 2)(x-1)^2 \cdot x^6 \cdot D_x^6 \; + \cdots \qquad (124)$$

Combining these rescalings with the shift of θ by $1/2$ one easily deduces self-adjoint operators, for instance:

$$E_2 \quad \longrightarrow \quad (2\theta + 1)^7 \; - 16x \cdot (16\theta^4 + 64\theta^3 + 112\theta^2 + 96\theta + 33)(\theta + 1)^3 + 16x^2 \cdot (\theta + 1)^2 \cdot (\theta + 2)^2 \cdot (2\theta + 3)^3.$$

From now on, let us consider the "rescaled" $\hat{E}_n$. The exterior squares of these order-seven operators $\hat{E}_n$ are of order 14 instead of the order 21 one could expect generically. The *exterior cube* of these order-seven operators are of order 27 (instead of order 35). The symmetric squares of these order-seven operators are of order 27, instead of the order 28 one could expect generically (see (125)).

Note that any operator such that its symmetric square is of order less than the generic expected order (here 28) is such that its solutions verifies a quadratic relation. For instance, the seven formal solutions of the order-seven operator $\hat{E}_1$ verifies the simple quadratic identity:

$$2S_1S_7 \; - 2S_6S_2 \; + 2S_5S_3 \; - S_4^2 \quad = \quad 0. \qquad (125)$$

Let us consider the operators $E_n^{(m)}$ non trivially homomorphic to the $\hat{E}_n$'s:

$$E_n^{(m)} \cdot D_x^m \quad = \quad L_m \cdot \hat{E}_n, \qquad (126)$$

where L_m is an order-m operator. For $m = 1$ the *exterior squares* of the $E_n^{(m)}$'s are of order 21 (as expected generically), but the symmetric squares of the $E_n^{(m)}$'s are still of order 27. The exterior cube of the $E_n^{(m)}$'s are of order 34 instead of the order 35 expected generically.

For $m = 2$ the symmetric squares of the $E_n^{(m)}$'s are still of order 27, however for $m = 3$ the symmetric squares of the $E_n^{(m)}$'s is of the order 28 expected generically. The *exterior cube* of the $E_n^{(m)}$'s are of order 35 expected generically.

The exterior squares of the $E_n^{(1)}$'s are actually a *direct sum* of an order-fourteen operator and an order-seven operator

$$Ext^2(E_n^{(1)}) \quad = \quad L_{14}^{(n)} \oplus L_7^{(n)}, \qquad (127)$$

where the order-seven operators are actually simply conjugated to the $\hat{E}_n$'s:

$$L_7^{(1)} \; = \; \frac{1}{(1-x)^{3/2}\,x^3} \cdot \hat{E}_1 \cdot (1-x)^{3/2}\,x^3, \qquad L_7^{(n)} \; = \; \frac{1}{(1-x)\,x^3} \cdot \hat{E}_n \cdot (1-x)\,x^3.$$

The *symmetric squares* of the $E_n^{(3)}$'s are actually a *direct sum* of an order-27 operator and an order-one operator

$$Sym^2(E_n^{(3)}) \quad = \quad L_{27}^{(n)} \oplus L_1^{(n)}, \qquad (128)$$

where the order-one operators $L_1^{(n)}$ have the following *rational* solutions r_n:

$$r_1 \; = \; \frac{1}{(x-1)^3 \cdot x^6}, \qquad r_n \; = \; \frac{1}{(x-1)^2 \cdot x^6} \qquad n \; = \; 2, \cdots, 5. \qquad (129)$$

The *exterior cubes* of the $E_n^{(2)}$'s are actually a *direct sum* of an order-27 operator $M_{27}^{(n)}$, an order-seven operator $M_7^{(n)}$ and an order-one $M_1^{(n)}$ operator

$$Ext^3(E_n^{(2)}) \quad = \quad M_{27}^{(n)} \oplus M_7^{(n)} \oplus M_1^{(n)}, \qquad (130)$$

where the order-one operators $M_1^{(n)}$ have an *algebraic* solution for $E_1^{(2)}$:

$$a_1 \;=\; \frac{1}{(x-1)^{9/2} \cdot x^9}, \qquad (131)$$

and the following *rational* solutions for the exterior cube of the other $E_n^{(2)}$'s:

$$\rho_n \;=\; \frac{1}{(x\,-1)^3 \cdot x^9}, \qquad\qquad n \;=\; 2, \cdots, 5. \qquad (132)$$

Remark 1: The emergence of the exceptional group G_2 corresponds to the appearance of rational (or square root of rational††) solutions for the *symmetric square and exterior cube* of these operators. This is reminiscent (see page 320 of Chapter 9 of [7]) of the (non-Fuchsian) order-seven operator $D_x^7 \,-x \cdot D_x \,-1/2$, which has the differential Galois group G_2, namely the *exceptional* subgroup [51] of $SO(7)$. If we had only rational (or square root of rational) solutions of the symmetric square of the operators we would have $SO(7)$ differential Galois groups: the appearance of the rational (or square root of rational) solutions for the *exterior cube* of these operators explains the emergence of this exceptional subgroup of $SO(7)$.

Remark 2: Throughout the paper, we see the systematic emergence of decompositions as *direct sums* (see for instance (127), (128), (130)) instead of just a factorization, each time we find a rational solution for some symmetric or exterior power. This should not be seen as a surprise. Indeed, the linear differential operator $E_n^{(m)}$ is *irreducible*. This implies that its differential Galois group is *reductive*, i.e. all its *representations are semi-simple* (i.e. decompose as a *direct sum of irreducible representations*): see section 2.2, specially discussion before lemma 2.3 in [32]. In practice, this means that if we perform any construction like Sym^m, Ext^r, etc, and if the corresponding operator factors, then it decomposes as an LCLM of *irreducible* operators (because the differential Galois group acts on the solution space of this operator, so the above applies).

Remark 3: All these results on symmetric squares, exterior squares and exterior cubes of (equivalent) order-seven operators have *not* been obtained using the Maple's DEtools command "ratsols" and "expsols", the corresponding algorithms being *not powerful enough* to cope with such examples of too large order. Furthermore, if one switches to differential systems representations, using the package [41] one finds†, again the corresponding algorithms‡ are *not powerful enough* to cope with such examples of too large order. One needs to go a step further, *switching to θ-systems*, a method that yields systematically simple poles.

†† In that case, the group is not connected but its Lie algebra is still $\mathfrak{g}_2$, i.e the connected component of the group which contains the identity is G_2.

† Use the commands with(TensorConstructions); with(IntegrableConnections); then companion-system(*), exterior-power-system(*,N), symmetric-power-system(*,N), RationalSolutions([*],[x]), HyperexponentialSolutions([*],[x]).

‡ We try to promote, in this paper the idea that switching to differential systems is a more intrinsic and powerful method that working on the operators (seen at first sight by physicists, as simpler). With these examples we see that even switching to differential systems is not enough: one needs to switch to θ-systems.

Seeking for rational solutions of differential systems (for regular systems like these), one (roughly) needs to find a transformation that cast them in simple poles. Then one finds the exponents, and then one reduces to polynomial solutions. Switching to θ-systems♯, one automatically has simple poles: this bypasses the first reduction step. However, when one considers the companion matrix, one needs to perform a reduction of each singularity: this can, in general, yield a lot of reductions.

Using the TensorConstruction package, the command to be used is Theta-companion-system or full-theta-companion-system. One needs to perform a "reduction at ∞" in order to find polynomial solutions. Doing all these tricks, one finally finds these results almost immediately for the symmetric squares, and for the exterior cubes.

5.1. Two-parameter and three-parameter deformations of $\hat{E}_1$

Another operator $\Omega(p,\, q)$ (generalizing $\hat{E}_1$), depending on *two* parameters p and q, is given in Appendix D (see also the operator L in section 5.2 of [30]).

Let us denote $\tilde{\Omega}(p,\, q)^{(m)}$ the order-seven linear differential operator homomorphic to $\Omega(p,\, q)$ with a D_x^m intertwiner (perform the LCLM of $\Omega(p,\, q)$ and D_x^m and rightdivide by D_x^m).

One has the following *direct sum* decompositions, *for arbitrary values* of p and q, for the *symmetric square* and *exterior cube* of these equivalent operators:

$$Sym^2(\tilde{\Omega}(p,\, q)^{(3)}) \;\; = \;\; L_{27} \oplus L_1, \tag{133}$$

and

$$Ext^3(\tilde{\Omega}(p,\, q)^{(2)}) \;\; = \;\; M_{27}^{(n)} \oplus M_7 \oplus M_1, \tag{134}$$

where the order-one operator L_1 has the rational solution $1/x^6/(x\,-1)^3$, and where M_1 has the square root of rational solution $1/x^9/(x\,-1)^{9/2}$. Furthermore one has

$$Ext^2(\tilde{\Omega}(p,\, q)^{(1)}) \;\; = \;\; L_{14} \oplus L_7, \qquad \text{where:} \tag{135}$$
$$L_7 \;=\; \frac{1}{(x\,-1)^{3/2}\cdot x^3}\cdot \Omega(p,\, q)\cdot\, (x\,-1)^{3/2}\cdot x^3.$$

We have the same results as (133), (134) and (135) for (the equivalent of) a simple one-parameter deformation of $\hat{E}_1$, namely:

$$\hat{E}_1(r) \;\; = \;\; \hat{E}_1 \;+\; r\cdot x\cdot (2\,\theta\,+1). \tag{136}$$

Note, however, that combining the previous deformation (136) on $\Omega(p,\, q)$ does yield a *three parameters* deformation satisfying direct sum decompositions like (133), (134) and (135)

$$\Omega(p,\, q,\, r) \;\; = \;\; \Omega(p,\, q) \;+ r\cdot x\cdot (2\,\theta\,+1), \qquad \text{where} \tag{137}$$

$$Sym^2(\tilde{\Omega}(p,\, q,\, r)^{(3)}) \;\; = \;\; L_{27} \oplus L_1, \tag{138}$$

$$Ext^3(\tilde{\Omega}(p,\, q,\, r)^{(2)}) \;\; = \;\; M_{27}^{(n)} \oplus M_7 \oplus M_1, \tag{139}$$

where L_1 and M_1 have respectively the solutions $1/x^6/(x-1)^3$ and $1/x^9/(x-1)^{9/2}$.

♯ An alternative way, if one does not want to switch to θ-systems, would be to perform a "Moser-reduction" on the companion matrix *before* calculating the symmetric or exterior powers (these powers preserving the order of the poles).

Furthermore one has

$$Ext^2(\tilde{\Omega}(p,\, q,\, r)^{(1)}) \;\; = \;\; L_{14} \oplus L_7, \qquad\qquad \text{where:} \qquad (140)$$
$$L_7 \; = \; \frac{1}{(x\,-1)^{3/2} \cdot x^3} \cdot \Omega(p,\, q,\, r) \cdot (x\,-1)^{3/2} \cdot x^3.$$

Remark: If one wants to get rid of algebraic solution like $1/x^9/(x\,-1)^{9/2}$, just perform a conjugation of $\hat{E}_1$: $\hat{E}_1 \; \rightarrow \; (x-1)^{-1/2} \cdot \hat{E}_1 \cdot (x-1)^{1/2}$, the rational solution of L_1 will become $1/x^6/(x-1)^4$ and the algebraic solution of M_1 will become $1/x^9/(x\,-1)^6$

This three-parameter conjugated operator† $(x-1)^{-1/2} \cdot \Omega(p,\, q,\, r) \cdot (x-1)^{1/2}$ probably also has the exceptional group $G_2(C)$ as its differential Galois group.

5.2. Three-parameter family of order-seven operators with exceptional Galois groups

The order-seven operators $\hat{E}_i$ for $i \;=\; 2 \cdots 5$ can also be seen as special cases of another order-seven operator $\Omega_{a,b,c}$ depending on three parameters (see Appendix D.2, see also operator P_1 in section 5.1 of [30]).

Let us denote again $\tilde{\Omega}^{(m)}_{a,b,c}$ the order-seven linear differential operator homomorphic to $\Omega_{a,b,c}$ with a D_x^m intertwiner (perform the LCLM of $\Omega_{a,b,c}$ and D_x^m and rightdivide by D_x^m). The operator $\Omega_{a,b,c}$ is generically irreducible. Again this implies *direct sum decompositions* for any construction Sym^m, Ext^r.

The *symmetric square* of $\tilde{\Omega}^{(3)}_{a,b,c}$ is actually a *direct sum* of an order-27 operator and an order-one operator, L_1, with the rational solution $1/(x-1)^2/x^6$

$$Sym^2(\tilde{\Omega}^{(3)}_{a,b,c}) \;\; = \;\; L_{27} \oplus L_1 \qquad (141)$$

and the *exterior cube* of $\tilde{\Omega}^{(2)}_{a,b,c}$ is actually a *direct sum* of an order-27 operator $M^{(n)}_{27}$, an order-seven operator $M^{(n)}_7$, and an order-one $M^{(n)}_1$ operator which has the rational solution $1/(x-1)^3/x^9$:

$$Ext^3(\tilde{\Omega}^{(2)}_{a,b,c}) \;\; = \;\; M^{(n)}_{27} \oplus M^{(n)}_7 \oplus M^{(n)}_1. \qquad (142)$$

Furthermore one has†

$$Ext^2(\tilde{\Omega}^{(1)}_{a,\,b,\,c}) \;\; = \;\; L_{14} \oplus L_7, \qquad\qquad \text{where:} \qquad (143)$$
$$L_7 \; = \; \frac{1}{(x\,-1) \cdot x^3} \cdot \Omega_{a,\,b,\,c} \cdot (x\,-1) \cdot x^3.$$

This three-parameter operator probably also has the exceptional group $G_2(C)$ as its differential Galois group.

6. Comments and speculations: diagonal of rational functions

Let us recall that the (minimal) linear differential operators for the $\chi^{(n)}$'s, the n-particle contributions of the magnetic susceptibility of the square Ising model, are not irreducible, but *factor into many irreducible operators* of various orders [11, 12, 13, 14, 52] (two, three, four, ...). For all the factors for which the calculations can

† For $\Omega(p,\, q,\, r)$ the group is the exceptional group $G_2(C)$ *up to a center*, as a consequence of the emergence of a square root $1/x^9/(x\,-1)^{9/2}$.

† Note that these results (141), (142), (143), are obtained for *arbitrary values of the three parameters* $a,\; b,\; c$, of $\Omega_{a,b,c}$ independently of the fact that $\Omega_{a,b,c}$ is MUM or not.

be performed§ we have seen that these irreducible factors are *actually homomorphic to their adjoint*. Thus, the interesting question is to see whether all the factors of these (minimal) operators for the $\chi^{(n)}$'s are homomorphic to their adjoint, i.e. have a "special" differential Galois group, possibly as a consequence of the fact that the $\chi^{(n)}$'s are *diagonals of rational functions* (see [18, 19] for a definition).

In this paper we underline selected linear differential operators having selected differential structures (special differential Galois groups) characterized in a differential algebra way (homomorphisms to their adjoint, rational, or hyperexponential [42], solutions of their exterior or symmetric powers). The idea is to disentangle these selected geometrical properties from other selected structures of a more arithmetic properties (globally bounded series solutions [18, 19]), both kinds of selected properties occurring simultaneously with the concept of "modularity". It is important to understand the relationship between these two kinds of properties. Operators with selected differential Galois groups do not necessarily correspond to globally bounded solution series [18, 19]. It is thus natural to see whether operators with globally bounded solution series [18, 19] necessarily correspond to selected differential Galois groups. This question being probably too difficult to address, let us ask the following question: if a linear differential operator has solutions that are *diagonals of rational functions*†, does it necessarily correspond to selected differential Galois groups, or, more simply, are such operators homomorphic to their adjoint (possibly in an algebraic extension) ? Note that we have accumulated a quite large number of operators with solutions that are *Hadamard products* [55, 56] of *algebraic functions* (and are thus simple examples of diagonals of rational functions [18, 19]). They all have been seen to be homomorphic to their adjoint (sometimes up to algebraic extensions). Let us recall that *diagonals of rational functions* are (most of the time transcendental) functions that are the *simplest extensions of algebraic functions* [18, 19] (modulo each prime, they are algebraic functions). It is worth noting that linear differential operators with algebraic solutions are always homomorphic to their adjoint (up to an algebraic extension). It is thus tempting to see whether (the factors of minimal) differential operators with solutions that are *diagonal of rational functions* are necessarily *homomorphic to their adjoint* (possibly in an algebraic extension). In order to get some hint on this question, we consider a set of simple but, hopefully, generic enough‡, diagonals of rational functions, find the minimal operators that annihilate them, and check wether the factors of these operators could all be homomorphic to their adjoint (up to algebraic extensions).

6.1. Diagonal of rational function: a heuristic simple example of an arbitrary number of variables

Let us first consider one of the simplest example of diagonal of rational functions of N variables, namely the diagonal of the rational function

$$S_N \;=\; Diag\Big(\frac{1}{1\,-x_1\,-x_2\;\cdots\,-x_N}\Big) \;=\; \sum_{k=0}^{\infty}\frac{(k\,N)!}{(k!)^N}\cdot x^k. \qquad (144)$$

§ There are factors of order 12 or 23, that are too large to see, by brute-force calculations, if these operators are homomorphic to their adjoint, or such that their exterior or symmetric square could have a rational solution.

† Diagonals of rational functions are necessarily solutions of linear differential operators (see Lipshitz [53, 54]).

‡ We try to avoid operators with hypergeometric or Hadamard product solutions.

The series S_N are solutions of the order $N-1$ linear differential operators L_{N-1}

$$L_{N-1} \; = \; \sum_{k=0}^{N-1} x^k \cdot (\alpha_k \cdot x \, - \, \beta_k) \cdot D_x^k \; = \; x^{N-2} \cdot (N^N \cdot x \, -1) \cdot D_x^{N-1} \qquad (145)$$
$$+ \, x^{N-3} \cdot \frac{N-1}{2} \cdot \Big((N-1) \cdot N^N \cdot x \, - \, (N-2) \Big) \cdot D_x^{N-2} \quad + \cdots$$

which, are remarkably, *self-adjoint* operators. The first operators read:

$$L_4 \; = \; (3125\, x - 1) \cdot x^3 \, D_x^4 \; + 2\,(12500\, x \, - 3) \cdot x^2 \, D_x^3 \; + (45000\, x \, - 7) \cdot x \, D_x^2$$
$$+ \, (15000\, x - 1) \cdot Dx \; + 120,$$
$$L_5 \; = \; (46656\, x \, - 1) \cdot x^4 \, D_x^5 \; + 10\,(58320\, x \, - 1) \cdot x^3 \, D_x^4 \; + 25\,(79056\, x \, - 1) \cdot x^2 \, D_x^3$$
$$+ \, 15\,(126360\, x \, - 1) \cdot x \, D_x^2 \; + (331920\, x \, - 1) \cdot Dx \; + 720,$$

or, more simply in $\theta \, = \, x \, D_x$:

$$x \cdot L_4 \; = \; 5\, x \cdot (5\,\theta + 1) \cdot (5\,\theta + 2) \cdot (5\,\theta + 3) \cdot (5\,\theta + 4) \, - \theta^4,$$
$$x \cdot L_5 \; = \; 6\, x \cdot (6\,\theta + 1) \cdot (6\,\theta + 2) \cdot (6\,\theta + 3) \cdot (6\,\theta + 4) \cdot (6\,\theta + 5) \, - \theta^5,$$
$$x \cdot L_6 \; = \; 7\, x \cdot (7\,\theta + 1) \cdot (7\,\theta + 2) \cdot (7\,\theta + 3) \cdot (7\,\theta + 4) \cdot (7\,\theta + 5) \cdot (7\,\theta + 6) \, - \theta^6,$$

and generally

$$x \cdot L_{N-1} \; = \; N \cdot x \cdot (N\,\theta + 1) \cdot (N\,\theta + 2) \; \cdots \; (N\,\theta + N - 1) \; - \theta^{N-1}, \qquad (146)$$

which makes crystal clear that these operators are hypergeometric operators with ${}_{N-1}F_{N-2}$ solutions. For instance, for L_4, we recover the ${}_4F_3$ hypergeometric solution occurring in Candelas et al. paper [57]:

$${}_4F_3\Big([\frac{1}{5}, \, \frac{2}{5}, \, \frac{3}{5}, \, \frac{4}{5}], \, [1, \, 1, \, 1], \; 5^5 \cdot x \Big). \qquad (147)$$

For arbitrary values of N we get, for L_{N-1}, the ${}_{N-1}F_{N-2}$ hypergeometric solution:

$${}_{N-1}F_{N-2}\Big([\frac{1}{N}, \, \frac{2}{N}, \, \cdots \, , \; \frac{N-1}{N}], \, [1, \, 1, \, \cdots \, , \; 1], \; N^N \cdot x \Big). \qquad (148)$$

It is worth noting, for larger values of N, that the L_{N-1} operators are such that, not only the series-solution, associated with (148), is a globally bounded series [18, 19] (with the N^N factor in the argument of the hypergeometric function it is even a series with *integer* coefficients), but that the series for the nome and *all the Yukawa couplings*, are all series with *integer* coefficients, thus corresponding to a *modularity* of the operators. These results can be seen to be a consequence of [58] which gives the special parameters of generalized hypergeometric equations leading to mirror maps with integral Taylor coefficients at $x \, = \, 0$. For instance, the nome $q(L_{N-1})$ of the first L_{N-1}'s read:

$$q(L_3) \; = \; x \; + 104\, x^2 \; + 15188\, x^3 \; + 2585184\, x^4 \; + 480222434\, x^5 \; + \; \cdots$$
$$q(L_4) \; = \; x \; + 770\, x^2 \; + 1014275\, x^3 \; + 1703916750\, x^4 \; + 3286569025625\, x^5 \; + \; \cdots$$
$$q(L_5) \; = \; x \; + 6264\, x^2 \; + 87103188\, x^3 \; + 1736438167584\, x^4$$
$$+ \, 42329034160944354\, x^5 \; + \; \cdots$$
$$q(L_6) \; = \; x \; + 56196\, x^2 \; + 9646450758\, x^3 \; + 2718983725393656\, x^4$$
$$+ \, 1002323538601613453169\, x^5 \; + \; \cdots$$

$$\begin{aligned}
q(L_7) \; &= \; x \, + 554112\, x^2 \, + 1360868807232\, x^3 + 6344223280197623808\, x^4 \\
& \qquad + 41288465594250793127633184\, x^5 \, + \, \cdots \\
q(L_8) \; &= \; x \, + 5973264\, x^2 \, + 240205268638728\, x^3 + 21222454448347058876544\, x^4 \\
& \qquad + 27811159193696216862379353195 24\, x^5 \, + \, \cdots \\
q(L_9) \; &= \; x \, + 69998400\, x^2 \, + 52035672968460000\, x^3 + 98009214813679052640000000\, x^4 \\
& \qquad + 2896912846893653458608921137437500 00\, x^5 \, + \, \cdots
\end{aligned} \tag{149}$$

and all the Yukawa series, including the "higher order Yukawa couplings†", K_n, are globally bounded, and, even, series with *integer coefficients*. We actually found the following relations between the Yukawa couplings:

$$K_4(L_4) \; = \; K_3(L_4)^2,$$

$$K_4(L_5) \; = \; K_3(L_5)^3, \qquad \text{and:} \qquad K_5(L_5) \; = \; K_3(L_5)^5,$$

$$K_5(L_6) \; = \; K_4(L_6)^2, \qquad \text{and:} \qquad K_6(L_6) \; = \; K_4(L_6)^3,$$

$$K_7(L_7) \; = \; \Bigl({{K_4(L_7)} \over {K_3(L_7)}}\Bigr)^7, \quad K_6(L_7) \; = \; \Bigl({{K_4(L_7)} \over {K_3(L_7)}}\Bigr)^5, \quad \text{and:} \quad K_5(L_7) \; = \; \Bigl({{K_4(L_7)} \over {K_3(L_7)}}\Bigr)^3,$$

$$K_8(L_8) \; = \; \Bigl({{K_5(L_8)} \over {K_3(L_8)}}\Bigr)^4, \quad K_7(L_8) \; = \; \Bigl({{K_5(L_8)} \over {K_3(L_8)}}\Bigr)^3 \quad \text{and:} \quad K_6(L_8) \; = \; \Bigl({{K_5(L_8)} \over {K_3(L_8)}}\Bigr)^2,$$

$$K_9(L_9) \; = \; \Bigl({{K_5(L_9)} \over {K_4(L_9)}}\Bigr)^9, \quad K_8(L_9) \; = \; \Bigl({{K_5(L_9)} \over {K_4(L_9)}}\Bigr)^7, \quad K_7(L_9) \; = \; \Bigl({{K_5(L_9)} \over {K_4(L_9)}}\Bigr)^5,$$

$$\text{and:} \qquad K_6(L_9) \; = \; K_3(L_9) \cdot \Bigl({{K_5(L_9)} \over {K_4(L_9)}}\Bigr)^3,$$

$$K_{10}(L_{10}) \; = \; \Bigl({{K_6(L_{10})} \over {K_4(L_{10})}}\Bigr)^5, \quad K_9(L_{10}) \; = \; \Bigl({{K_6(L_{10})} \over {K_4(L_{10})}}\Bigr)^4, \quad K_8(L_{10}) \; = \; \Bigl({{K_6(L_{10})} \over {K_4(L_{10})}}\Bigr)^3,$$

$$\text{and:} \qquad K_7(L_{10}) \; = \; K_3(L_{10}) \cdot \Bigl({{K_6(L_{10})} \over {K_4(L_{10})}}\Bigr)^2,$$

where these Yukawa couplings read respectively

$$\begin{aligned}
K_3(L_4) \; &= \; 1 \, + 575\, x \, + 1418125\, x^2 \, + 3798200625\, x^3 \, + 10597067934375\, x^4 \\
& \quad + 30287765070550575\, x^5 + \, \cdots,
\end{aligned}$$

$$\begin{aligned}
K_3(L_5) \; &= \; 1 \, + 10080\, x + 357073920\, x^2 + 13943124679680\, x^3 \\
& \quad + 570470634728386560\, x^4 + 23986351416805190461440\, x^5 + \, \cdots
\end{aligned}$$

For L_6 and L_7 one has *two* independent Yukawa couplings. For L_8 and L_9 one has *three* independent Yukawa couplings, for L_{10} and L_{11} one has *four* independent Yukawa couplings, ... These series are displayed in Appendix E.

For N an *odd* integer the *exterior square* of the $(N-1)$-order operator (145) is of order $(N-1)\,(N-2)/2 \, - 1$ instead of $(N-1)\,(N-2)/2$. For N an *even* integer the *symmetric square* of the $(N-1)$-order operator (145) is of order $N\,(N-1)/2 - 1$ instead of $N\,(N-1)/2$. Similarly to (65) or (89), introducing an equivalent operator $\tilde{L}^n_{N-1}$, such that $S^n \cdot L_{N-1} \, = \, \tilde{L}^n_{N-1} \cdot D_x^n$, the exterior or symmetric square of that equivalent operator has, for a well-suited value of n the rational solution $1/x^{N-2}/(N^N\, x \, - 1)$.

† See appendic C, and especially C.2, in [18, 19] for the definition of these "higher order" Yukawa couplings K_n.

6.2. Diagonal of rational function: heuristic simple examples of three variables

Increasing the degree of the rational functions, another example corresponds to the diagonal of

$$R(x,\, y,\, z) \;\; = \;\; \frac{1 \, - x \, + \, y\, z}{1 \, - 3\, x\, z \, - 5\, y^2}. \qquad (150)$$

Unfortunately this diagonal is solution of an order-two operator with *algebraic solutions*, $(4 - 1215\, x^2) \cdot D_x^2 - 3645 \cdot x\, D_x - 1080$ (which is conjugated to its adjoint).

We have a similar algebraic result with

$$R(x,\, y,\, z) \;\; = \;\; \frac{1 \, - 7\, x \, + \, 2\, y\, z}{1 \, + 3\, x\, z \, - 5\, y^3}. \qquad (151)$$

Its diagonal is the lacunary series

$$1 \;\; - 540\, x^3 \; + 510300\, x^6 \; - 541282500\, x^9 \; + 604514137500\, x^{12}$$
$$- \, 695204544150000\, x^{15} \, + 814769502147562500\, x^{18} \; + \; \cdots \qquad (152)$$

which is an *algebraic function* solution of an order-three operator (homomorphic to its adjoint).

Another example corresponds to the diagonal of

$$R(x,\, y,\, z) \;\; = \;\; \frac{1 \, - x}{1 \, - 3\, x \, + z \, - 5\, y^2}. \qquad (153)$$

which also corresponds to the series expansion of a hypergeometric function:

$$\frac{2}{15} \; + \; \frac{13}{15} \cdot {}_4F_3\Big([\frac{1}{5}, \, \frac{2}{5}, \, \frac{3}{5}, \, \frac{4}{5}], \; [\frac{1}{2}, \, \frac{1}{2}, \, 1], \;\; 5^6 \cdot (\frac{3}{4} \cdot x)^2\Big)$$
$$= \;\; 1 \;\, + 1170\, x^2 \; + 5528250\, x^4 \; + 33202669500\, x^6 \qquad (154)$$
$$+ \, 221602408706250\, x^8 \; + 1569831463275075000\, x^{10} \; + \; \cdots$$

The corresponding order-five operator is a *direct sum* $D_x \oplus M_4$ where the order-four operator M_4, which annihilates the ${}_4F_3$ in (154), is homomorphic to its adjoint and such that its exterior square has a rational solution namely $1/x/(9 \cdot 5^6\, x^2 - 16)$. The order-four operator M_4 is of the form (100), namely:

$$M_4 \;\; = \;\; L_2 \cdot c_0(x) \cdot M_2 \, + \, \frac{9}{5^4} \cdot \frac{1}{c_0(x)}, \qquad\qquad c_0(x) \;\; = \;\; x^2 \cdot \Big(x^2 - \frac{16}{9 \cdot 5^6}\Big), \qquad (155)$$

where L_2 and M_2 are two order-two *self-adjoint* operators‡.

Noting that the diagonal of

$$\tilde{R}(x,\, y,\, z) \;\; = \;\; \frac{1}{1 \, - 3\, x \, + z \, - 5\, y^2}. \qquad (156)$$

is solution of the order-four operator M_4, and is nothing but

$${}_4F_3\Big([\frac{1}{5}, \, \frac{2}{5}, \, \frac{3}{5}, \, \frac{4}{5}], \; [\frac{1}{2}, \, \frac{1}{2}, \, 1], \;\; \frac{9 \cdot 5^6}{16} \cdot x^2\Big), \qquad (157)$$

and introducing $\hat{R}(x,\, y,\, z) \, = \, 15\, R(x,\, y,\, z) \, - 13\, \tilde{R}(x,\, y,\, z) \, - 2$, one easily deduces, from (157), that the diagonal of

$$\hat{R}(x,\, y,\, z) \;\; = \;\; \frac{10\, y^2 \, - 9\, x \, - 2\, z}{1 \, - 3\, x \, + z \, - 5\, y^2} \qquad \text{or} \qquad \frac{2 \, - 15\, x}{1 \, - 3\, x \, + z \, - 5\, y^2}, \qquad (158)$$

is equal to zero.

These cases, reducible to algebraic or hypergeometric situations, are still too simple to be representative of the "generic" situation.

‡ M_2 is the product of two order-one operators the right factor having the polynomial solution $x^2\, (9 \cdot 5^6\, x^2 \, - 16)$.

6.3. Towards a "generic" diagonal of rational function example

Trying to avoid these too simple cases§ reducible to hypergeometric functions (or Hadamard product of algebraic functions), we have considered the operator annihilating the diagonal of a rational function of three variables, hopefully involved enough, with no symmetry between the three variables, to be seen as a "generic" diagonal of a rational function.

6.3.1. Towards a "generic" diagonal of rational function: a first example

The rational function we have considered reads:

$$R(x,\, y,\, z) \;=\; \frac{1}{1\,-3\,x\,-5\,y\,-7\,z\,+x\,y\,+2\,y\,z^2\,+3\,x^2\,z^2}. \tag{159}$$

The diagonal of this rational function reads‡:

$$\begin{aligned} S_0^{(0)} \;=\;\; & Diag(R(x,\, y,\, z)) \;=\; 1\; +616\,x\; +947175\,x^2\; +\, 1812651820\,x^3 \\ & +\, 3833011883965\,x^4 \,+\, 8582819380142616\,x^5 \,+\, 19946071353510410136\,x^6 \\ & +\, 47578122531207001944168\,x^7 \,+\, 115702070514540009854741415\,x^8 \\ & +\, 285583642613093627090885877280\,x^9 \\ & +\, 713269435359072253352128013072035\,x^{10} \;+\; \cdots \end{aligned} \tag{160}$$

The minimal order operator that annihilates the diagonal of this rational function (165) is a *quite large order-six* linear differential operator†. Again, this operator is too large to check that it is *homomorphic to its adjoint*. We can, however, check that its *exterior square* is of order 15. However, switching to the associated differential theta-system, we have been able to see that it is actually *homomorphic to its adjoint*: one actually finds the *exterior square* of the associated differential system has a rational solution (but not its symmetric square). The differential Galois group thus corresponds to a *symplectic structure*.

In fact this operator is *not MUM*. It has four solutions, analytic at $x \,=\, 0$, namely $S_0^{(0)}$ given by (160) and

$$\begin{aligned} S_0^{(1)} \;=\;\; & x \;-\, \frac{947569825302083891091227422045}{31916864416389315849900085 14}\,x^4 \\ & \quad -\, \frac{1303834451394235031575824909127468 8499}{196260345614646390862796723535 32}\,x^5 \;+\; \cdots, \\ S_0^{(2)} \;=\;\; & x^2 \;+\frac{60}{7^3}\,x^4 \;-\, \frac{576}{7^4}\,x^5 \;+\; \cdots, \\ S_0^{(3)} \;=\;\; & x^3 \;+\frac{30608172563777847511388970395}{1447476844280694596367350 8}\,x^4 \\ & \quad +\, \frac{66377383028880230017305650111795446 51}{14018596115331885061628337395 38}\,x^5 \;+\; \cdots \end{aligned} \tag{161}$$

§ When the operators annihilating diagonal of rational functions are of order two, one often finds modular forms, the corresponding nome being seen to be a globally bounded series [18, 19]. A set of examples of diagonal of Szego's rational functions can be found in [59].

‡ Use the maple command mtaylor(F, [x,y,z], terms), to get the series in three variables, then take the diagonal. Other method, in Mathematica install the risc package Riscergosum [60], and in HolonomicFunctions' use the command FindCreativeTelescoping.

† We thank Alin Bostan for providing this order six operator from a creative telescopic code (not by guessing).

the last series $S_0^{(3)}$ being *not globally bounded*. The two other solutions have a log (but no log^2, log^3, ...):

$$S_1^{(0)} \, = \, S_0^{(0)} \cdot \ln(x) \, + \, T_0^{(0)}, \qquad S_1^{(2)} \, = \, S_0^{(2)} \cdot \ln(x) \, + \, T_0^{(2)}, \tag{162}$$

the two series $T_0^{(0)}$ and $T_0^{(2)}$ being analytic at $x \, = \, 0$, for instance:

$$T_0^{(0)} \, = \, \frac{1769904090259426475015551868948047756831494229112489}{634749357269938082528445401418795584280 0} \, x^4 \tag{163}$$
$$+ \, \frac{215779837076611177067085144369886918584316327157449735 27227853}{2134044159999499486819820443373128352432343420 0} \, x^5 \, + \, \cdots$$

Of course there is an ambiguity in all these solutions, except $S_0^{(3)}$ which is well defined: one can add $S_0^{(3)}$ to $S_0^{(2)}$, etc ... There is an ambiguity on $S_0^{(0)}$, and $S_0^{(2)}$, but the match of $S_0^{(0)}$ diagonal of the rational function and the form $S_1^{(2)} \, = \, S_0^{(2)} \cdot \ln(x) \, + \, T_0^{(2)}$ fixes the normalization of $S_0^{(0)}$, and $S_0^{(2)}$. Thus $T_0^{(0)}$ is defined by (163), up to the solutions $S_0^{(0)}$, $S_0^{(1)}$ and $S_0^{(2)}$. Trying to go further the fact that the operator is not MUM, one can try to define two nomes by

$$q_1 \, = \, \exp\Bigl(\frac{S_1^{(0)}}{S_0^{(0)}}\Bigr), \qquad q_2 \, = \, \exp\Bigl(\frac{S_1^{(2)}}{S_0^{(2)}}\Bigr), \tag{164}$$

and, using this ambiguity, seek for $T_0^{(0)}$ and $T_0^{(2)}$ such that the two nomes are globally bounded series. Unfortunately, it is almost impossible to see if one can build nomes, such that their series are globally bounded series [18, 19].

With this example that is not MUM, we exclude any simple modularity property for the operator, where the series for the nome, Yukawa couplings, etc ... would be globally bounded. Diagonal of rational functions do not necessarily yield modularity.

6.3.2. Towards a "generic" diagonal of rational function: a second example

Let us consider another simpler example with the diagonal of another rational function of three variables:

$$R(x, \, y, \, z) \, = \, \frac{1}{1 \, + \, z \, - \, x\,y \, + \, x\,z \, + \, y^2}. \tag{165}$$

The diagonal of this rational function reads:

$$S_0^{(0)} \, = \, Diag(R(x, \, y, \, z)) \, = \, 1 \, - 2\,x \, + 3\,x^2 \, + 40\,x^3 \, - 545\,x^4 \, + 3948\,x^5$$
$$- \, 14910\,x^6 \, - 55176\,x^7 \, + 1544895\,x^8 \, - 14999270\,x^9 \, + 82528303\,x^{10}$$
$$- \, 29585712\,x^{11} \, - 5093494406\,x^{12} \, + \, \cdots \tag{166}$$

It is solution of an irreducible order-four operator L_4 that is *not* MUM. This operator is non-trivially *homomorphic to its adjoint*, with order-two intertwiners and is actually of the form (100):

$$L_4 \, = \, L_2 \cdot c_0(x) \cdot M_2 \, + \, \frac{\lambda}{c_0(x)}, \quad \lambda \, = \, \Bigl(\frac{26}{639}\Bigr)^2, \quad c_0(x) \, = \, \frac{\lambda}{9} \cdot \frac{p_3(x)^2 \cdot p_4(x)}{p_6(x)},$$
$$p_6(x) \, = \, 676\,x^6 \, + 10514\,x^5 \, - 2047\,x^4 \, + 82424\,x^3 \, - 15796\,x^2 \, + 10304\,x \, + 448,$$
$$p_4(x) \, = \, 729\,x^4 - 1568\,x^3 + 984\,x^2 + 192\,x + 16, \quad p_3(x) \, = \, (71\,x + 14)\,(x - 2) \cdot x,$$

where L_2 and M_2 are two *self-adjoint* operators, their Wronskian reading respectively

$$\frac{x^2 \cdot p_3(x) \cdot p_4(x)^2}{p_6(x)}, \qquad \frac{p_3(x)}{x^2 \cdot p_4(x)}. \tag{167}$$

The *exterior square* of L_4 is an order-six operator with a *rational function solution* $R(x)$, corresponding to the direct sum decomposition:

$$Ext^2(L_4) \;=\; L_5 \oplus \Big(D_x \,-\, \frac{d\ln(R(x))}{dx}\Big), \qquad R(x) \;=\; \frac{p_3(x)}{x^2 \cdot p_4(x)}, \qquad (168)$$

where L_5 is an irreducible order-five operator. The operator L_4 has the following four solutions: the diagonal $S_0^{(0)}$ (see (166)), an analytic solution $S_0^{(1)}$

$$S_0^{(1)} \;=\; x \,-\, \frac{35}{2^2}\,x^2 \,+\, \frac{3185}{2^6}\,x^3 \,-\, \frac{35035}{2^8}\,x^4 \,-\, \frac{16207191}{2^{14}}\,x^5 \,+\, \frac{1217957741}{2^{16}}\,x^6$$
$$-\, \frac{165312417127}{2^{20}}\,x^7 \,+\, \frac{3091190741925}{2^{22}}\,x^8 \,+\, \frac{1071079996954825}{2^{30}}\,x^9 \,+\, \cdots \qquad (169)$$

and two formal series solution with a log, namely $S_1^{(0)} \,+\, \ln(x) \cdot S_0^{(0)}$ and also $S_1^{(1)} \,+\, \ln(x) \cdot S_0^{(1)}$

$$S_1^{(0)} \;=\; 2 \,-\, \frac{35}{2}\,x \,+\, \frac{5703}{56}\,x^2 \,-\, \frac{321597}{896}\,x^3 \,-\, \frac{2659681}{3584}\,x^4 \,+\, \frac{29836311703}{1146880}\,x^5$$
$$-\, \frac{1156839045933}{4587520}\,x^6 \,+\, \frac{722563886554257}{513802240}\,x^7 \,-\, \frac{550307089986855}{411041792}\,x^8$$
$$-\, \frac{22561115451957783769}{315680096256}\,x^9 \,+\, \cdots$$

and:

$$S_1^{(1)} \;=\; \frac{11}{2}\,x^2 \,-\, \frac{30163}{576}\,x^3 \,+\, \frac{792323}{2304}\,x^4 \,-\, \frac{219473079}{163840}\,x^5 \,-\, \frac{16188947647}{5898240}\,x^6$$
$$+\, \frac{70828996802681}{660602880}\,x^7 \,-\, \frac{27874560487367}{25165824}\,x^8 \,+\, \frac{5457381128456532577}{811748818944}\,x^9 \,+\, \cdots$$

If one introduces the nome, its series expansion does not seem to be globally bounded [18, 19]:

$$q(L_4) \;=\; x \cdot exp\Big(\frac{S_1^{(1)}}{S_0^{(1)}}\Big) \;=\; x \,+\, \frac{11}{2}\,x^2 \,+\, \frac{6269}{576}\,x^3 \,+\, \frac{43165}{1152}\,x^4 \,+\, \frac{1040535941}{13271040}\,x^5$$
$$-\, \frac{11364935021}{26542080}\,x^6 \,+\, \frac{851517278314609}{160526499840}\,x^7 \,-\, \frac{1854100924158503}{64210599936}\,x^8$$
$$+\, \frac{7900344144708245867787}{14794122225254400}\,x^9 \,+\, \cdots$$

Do note that the second analytic series is actually globally bounded. With a rescaling $x \,\rightarrow\, 2^4\,x$, the series (169) becomes a series with integer coefficients:

$$16\,x \,-\, 2240\,x^2 \,+\, 203840\,x^3 \,-\, 8968960\,x^4 \,-\, 1037260224\,x^5 \,+\, 311797181696\,x^6$$
$$-\, 42319978784512\,x^7 \,+\, 3165379319731200\,x^8 \,+\, \cdots \qquad (170)$$

6.3.3. Towards a "generic" diagonal of rational function: a third example

Let us consider another example with the diagonal of another rational function of three variables:

$$R(x,\, y,\, z) \;=\; \frac{1 \,-\, x \,-\, y \,+\, x\,y\,z}{1 \,-\, x \,-\, y \,-\, x\,y \,-\, y^2\,z^3}. \qquad (171)$$

The diagonal of this rational function reads:

$$S_0^{(0)} \;=\; Diag(R(x,\, y,\, z)) \;=\; 1 \,+ x \,+ 10\, x^3 \,+ 32\, x^4 \,+ 966\, x^6 \,+ 3192\, x^7 \\ + 120340\, x^9 \,+ 401720\, x^{10} \,+ 16712150\, x^{12} + 56066920\, x^{13} \\ + 2466298800\, x^{15} + 8298484992\, x^{16} + 378403867380\, x^{18} \\ + 1275714885984\, x^{19} + 59651272137600\, x^{21} \;+\; \cdots \tag{172}$$

It is solution of an *order-five* operator L_5 which factors as $L_5 = L_4 \cdot D_x$, where L_4 is an *irreducible* order-four operator that is *not MUM*. The *exterior square* of L_4 is an order-six operator with a *rational function solution* $R(x)$, corresponding to the *direct sum* decomposition:

$$Ext^2(L_4) \;=\; L_5 \,\oplus\, \Bigl(D_x \,-\, \frac{d\ln(R(x))}{dx}\Bigr), \qquad R(x) \;=\; \frac{p_{10}(x)}{x^2 \cdot\, p_6(x)^2},$$

$$p_{10}(x) \;=\; 11008\, x^{10} + 165760\, x^9 - 637392\, x^8 + 383388\, x^7 + 196287\, x^6 \\ - 281004\, x^5 - 66582\, x^4 - 45360\, x^3 + 15660\, x^2 - 810\, x + 162,$$

$$p_6(x) \;=\; 1024\, x^6 - 9909\, x^3 + 54. \tag{173}$$

This order-four operator L_4 is non-trivially homomorphic to its adjoint, with order-two intertwiners and is actually of the form (100):

$$L_4 \;=\; L_2 \cdot c_0(x) \cdot M_2 \;+\; \frac{1305}{29584} \cdot \frac{1}{c_0(x)}, \qquad c_0(x) \;=\; \frac{145}{473344} \cdot \frac{p_{10}(x)^2}{p_{16}(x)},$$

$$p_{16}(x) \;=\; 37120\, x^{16} + 1255680\, x^{15} - 48870560\, x^{14} + 594756560\, x^{13} - 31084335\, x^{12} \\ + 2785358960\, x^{11} + 4430975954\, x^{10} - 8858296096\, x^9 - 1107376429\, x^8 \\ - 369545240\, x^7 + 4215494304\, x^6 - 1487095128\, x^5 - 466418052\, x^4 \\ + 228523680\, x^3 - 21096612\, x^2 + 2737800\, x - 717336,$$

where L_2 and M_2 are two self-adjoint operators, their Wronskian reading respectively

$$\frac{x^2 \cdot\, p_{10}(x) \cdot p_6(x)^2}{p_{16}(x)}, \qquad \frac{p_{10}(x)}{x^2 \cdot\, p_6(x)^2}. \tag{174}$$

The operator L_4 is not MUM: it has *three* solution analytic at $x = 0$, namely the derivative of diagonal (172)

$$\frac{dS_0^{(0)}}{dx} \;=\; 1 \,+ 30\, x^2 \,+ 128\, x^3 \,+ 5796\, x^5 \,+ 22344\, x^6 \,+ 1083060\, x^8 \;+\; \cdots, \tag{175}$$

and the two series solutions

$$x \;-\frac{595}{1107}\, x^2 \;+\frac{3515617}{1225449}\, x^3 \;+\frac{227188435}{1225449}\, x^4 \;-\frac{2520602}{15129}\, x^5 \;+\frac{8346429274}{11029041}\, x^6 \\ +\frac{2633989297550}{77203287}\, x^7 \;-\frac{42751323143}{1225449}\, x^8 \;+\; \cdots,$$

$$x^2 \;-\frac{595}{1107}\, x^3 \;+\frac{4523}{1107}\, x^4 \;+\frac{37758}{205}\, x^5 \;-\frac{1412590}{9963}\, x^6 \;+\frac{63250564}{69741}\, x^7 \\ +\frac{187516948}{5535}\, x^8 \;+\; \cdots, \tag{176}$$

One also has a formal series solution with a log, namely

$$S_1(x) \;+\; \ln(x) \cdot \frac{dS_0^{(0)}}{dx}, \tag{177}$$

where $S_1(x)$ is a series analytic at $x = 0$:

$$S_1(x) = \frac{1}{5\,x} + \frac{3083}{2214} + \frac{5222887}{4084830}\,x + \frac{956031447781}{22609534050}\,x^2 + \frac{661652916345161}{2502875419335}\,x^3 + \frac{2061248440531939}{12514377096675}\,x^4 + \frac{25090312744949777}{3089969653500}\,x^5 + \frac{4302952949201500235 9}{901035150960600}\,x^6 + \frac{21734062670504361827}{788405757090525}\,x^7 + \frac{5314508101399026611659}{3504025587069000}\,x^8 + \cdots \qquad (178)$$

The last series in (176), as well as $S_1(x)$ (see (178)), are *not globally bounded* series [18, 19].

Remark 1: The series (176) is, of course, also a diagonal of a rational function:

$$x \cdot \frac{dS_0^{(0)}}{dx} = Diag\Big(x \cdot \frac{\partial R(x,\, y,\, z)}{\partial x}\Big) \qquad (179)$$
$$= Diag\Big(y \cdot \frac{\partial R(x,\, y,\, z)}{\partial y}\Big) = Diag\Big(z \cdot \frac{\partial R(x,\, y,\, z)}{\partial z}\Big).$$

Remark 2: With this order-five example, we see that the minimal order operator L_5, that annihilates the diagonal of a rational function, is not necessarily irreducible. Let us recall the results of [18, 19] where we have shown that the $\tilde{\chi}^{(n)}$'s of the susceptibility of the square Ising model are *actually diagonals of rational functions*. The corresponding (globally nilpotent) linear differential operators annihilating the $\tilde{\chi}^{(n)}$'s are not irreducible, on the contrary they factor into many linear differential operators, of various orders [8, 11, 12, 13, 14, 17] (one, two, three, ..., 12, 23, ...). The interesting property we must focus on, is not that the (minimal order) linear differential operators annihilating the $\tilde{\chi}^{(n)}$'s are homomorphic to their adjoint, but that *all their factors* could be homomorphic to their adjoint. It is the differential Galois group of *all these factors* that we expect to be "special".

To sum up: One may consider the following conjecture: all the irreducible factors of the minimal order linear differential operator annihilating a diagonal of a rational function should be homomorphic to their adjoint (possibly on an algebraic extension).

Remark 3: Let us recall that the series of the hypergeometric function considered in [18, 19]

$${}_3F_2\Big([\frac{1}{9},\, \frac{4}{9},\, \frac{5}{9}],\, [\frac{1}{3},\, 1],\, 729\,x\Big) = 1 + 60\,x + 20475\,x^2 + 9373650\,x^3 + \cdots \qquad (180)$$

still remains a series with *integer* coefficients such that one cannot prove that it is the diagonal of a rational function, or discard that option. The minimal order operator annihilating this series is an order-three operator L_3 which is *not*‡ homomorphic to its adjoint.

If our conjecture above was correct, this would be a way to show that the series (180) *cannot be the diagonal of a rational function*.

7. Conclusion

Selected differential Galois groups correspond to symmetric square or exterior square, and possibly higher powers (as seen with order-seven operators with exceptional

‡ It is not even homomorphic up to algebraic extensions. The order-two intertwining operator M_2 such that $M_2 \cdot L_3 = adjoint(L_3) \cdot adjoint(M_2)$ has *transcendental* coefficients.

differential Galois groups of section (5)) of operators, or equivalent operators, having rational solutions (or N-th root of rational solutions, i.e. hyperexponential [24, 42] solutions). We have focused, in this paper, on a concept of "Special Geometry" corresponding to operators *homomorphic to their adjoint*. In a forthcoming publication, more focused on *differential systems*, and on demonstrations, we will show the equivalence of the homomorphism of an operator with its adjoint (possibly with algebraic extension), and the fact that its symmetric, or exterior, square of the corresponding differential systems have a rational (resp. N-th root of rational) solution.

In [22, 29] Bogner has been able, from the very existence of underlying Calabi-Yau varieties, to show that the Calabi-Yau differential operators are actually conjugated to their adjoint (Poincaré pairing). If one does not assume strong hypotheses like this one, it is not simple to disentangle the differential algebra structures (corresponding to selected differential Galois groups) we have addressed in this paper, and more "arithmetic" concepts associated to the notion of *modularity*, and the *integrality*, or *globally bounded* properties [18, 19] of the various series occurring with these differential operators (solution of the operator, the nome, the Yukawa couplings, ...). Recalling section 6, it is clear that the concept of "Special Geometry", which we address in this paper, does not necessarily yield‡ arithmetic properties like the globally bounded [18, 19] character of various series associated with the operators. Conversely, we know that globally bounded series [18, 19] do not necessarily correspond to holonomic functions (see the example of the *non-holonomic* susceptibility of the Ising model and its series with *integer* coefficients [52]). Along such "modularity" line, the idea that operators annihilating *diagonals of rational functions* should always correspond to a modularity property that the corresponding nome and all the Yukawa's [18, 19] are globally bounded series, has been ruled out (see, for instance, the example of subsection 6.3.1). From a mathematics viewpoint, there is still a lot of work to be performed to clarify the relations between these various neighboring concepts around the notion of "modularity". Along this line, it is probably useful to keep in mind all the simple examples† of section 6. From a physics viewpoint, one would like to identify, more specifically, what kind of "Special Geometry" we encounter (Calabi-Yau, selected hypergeometric functions up to pull-backs [20], ...).

In this paper, the emergence of selected differential Galois groups has been seen, in a down-to-earth physicist's viewpoint, as *differential algebra* properties: one calculates various exterior, or symmetric, powers, and looks (up to operator equivalence) for their rational solution (or hyperexponential [24, 42] solution), and *one calculates the homomorphisms of an operator with its adjoint*. We have shown that quite involved lattice Green operators of order six and eight are non trivially homomorphic to their adjoint, and that this yields the non trivial decompositions (60) and (83), where their intertwiners emerge in a crystal clear way (see also (114) in section 4.5). Such decompositions enable to understand why the lattice Green operator (47) has a differential Galois group included in the orthogonal group $O(6, \mathbb{C})$ instead of the symplectic $Sp(6, \mathbb{C})$ differential Galois group, that one might expect for an order-six operator: the intertwiners are of *odd orders*.

‡ Katz's book [7] provides examples of *self-adjoint* operators with special differential Galois groups that are *not even Fuchsian* (see also one of our first (hypergeometric) examples (2).

† For instance the order-six and order-eight operators G_6^{5Dfcc} and G_8^{6Dfcc} of sections 3.8 and 3.9 are not MUM.

Decompositions such as (90), (91) can be generalized for linear differential operators of *any even* order. In fact, one can actually use the decompositions (90), (91) as an *ansatz* to provide linear differential operators of *any even* order, that will *automatically* have selected differential Galois groups.

With these lattice Green operators, we see that the simple generalization of the *Calabi-Yau condition* (94) for operators of order $N > 4$ (namely the condition that their exterior square is of order less than the generic $N \cdot (N-1)/2$ order), is a *too restrictive concept for physics*. These lattice Green operators do not satisfy such higher-order generalization of the Calabi-Yau condition (94), but must be seen as higher-order generalization of a "weak Calabi-Yau condition" (see section 4.1) which amounts to saying that their exterior or symmetric squares have rational solutions, and that they are non-trivially homomorphic to their adjoint.

For order-four operators, Calabi-Yau operators are defined, among several other conditions (see Almkvist et al. [5]), essentially by the *Calabi-Yau condition* (94). It is, however, quite clear that any equivalent operator (in the sense of the equivalence of operator, i.e. homomorphic to the Calabi-Yau operator), is also a selected operator interesting for physics. We have shown, in this paper, that any order-four operator, non-trivially homomorphic to an irreducible operator satisfying the *Calabi-Yau condition* (94), has the following properties: it is *homomorphic to its adjoint* with *order-two* intertwiners, it has a simple decomposition (99), and it is such that its exterior square necessarily has a rational solution. Conversely, showing that "irreducible order-four operators such that their exterior square have a rational solution, or, even, have a decomposition (99)" are necessarily equivalent to irreducible operators satisfying the *Calabi-Yau condition* (94) is a difficult question.

To illustrate the differential algebra structures corresponding to higher-order symmetric or exterior powers, we have also analysed some families of order-seven self-adjoint operators with exceptional differential Galois groups, where one sees, very clearly, the emergence of rational solutions for symmetric square and *exterior cube* of equivalent operators. Finally, since among the Derived From Geometry n-fold integrals ("Periods") occurring in physics, we have seen that they are quite often *diagonals of rational functions* [18, 19], we have also addressed many examples of (minimal order) operators annihilating diagonals of rational functions, and remarked that they have *irreducible factors homomorphic to their adjoint*.

The n-fold integrals we encounter in theoretical physics are solutions of *Picard-Fuchs* differential equations, or in a more modern mathematical language [61, 62], variation of Hodge structures† and Gauss-Manin systems [30, 22, 63, 64]. According to mathematicians one should necessarily have for such variation of Hodge structures, a "*polarization*"‡ necessarily yielding to a "*duality*" which would send differential operators into their adjoint§. In our physical examples one seems to systematically inherit this "duality" on *each factor* of the minimal order operator, each irreducible factor being homomorphic to its adjoint. Along this line, section 6 strongly suggests to consider the conjecture that *(minimal) operators annihilating diagonal of rational functions solutions, always factor into irreducible operators homomorphic to their*

† Corresponding to the integrands in the n-fold integrals, namely one-parameter families of algebraic varieties.

‡ Which is a non-degenerated bilinear map (dual to an intersection mapping, see also the Poincaré duality [65, 66]).

§ The Picard-Fuchs linear differential operators associated with a family of smooth projective manifolds are homomorphic to their adjoint. This can be seen using the Poincaré duality [68].

adjoint, may be on algebraic extensions (these factors thus corresponding to "special" differential Galois groups).

This paper tries to promote the idea that, before deciphering the obfuscation of mathematicians on this subject, physicists should, in a down-to-earth way, use all the differential algebra tools†† they have at their disposal, checking systematically if the linear differential operators they work on, have factors which are homomorphic to their adjoint, or are such that, up to operator equivalence, their exterior (resp. symmetric) square have a rational solution. The emergence of this "duality" on all the irreducible factors of a large class of differential operators of physics needs to be understood.

Acknowledgments: We thank A. Bostan, G. Christol and P. Lairez for fruitful discussions on diagonals of rational functions. We thank Y. André, D. Bertrand, F. Beukers, L. Di Vizio, C. Voisin, for fruitful discussions on differential Galois groups and (self-adjoint) dualities in geometry. We also thank P. Boalch for fruitful isomonodromic discussions. As far as physicists authors are concerned, this work has been performed without any support of the ANR, the ERC or the MAE.

Appendix A. A decomposition of operators equivalent to operators satisfying the Calabi-Yau condition

Let us again consider an order-four operator Ω_4 which satisfies the Calabi-Yau condition (94), of Wronskian $w(x) \, = \, u(x)^2$.

Let us consider a monic order-four operator $\tilde{\Omega}_4$ which is (non-trivially) equivalent to the order-four operator Ω_4 satisfying the Calabi-Yau condition (94). This amounts to saying that there exist two (at most) order-three intertwiners U_3 and L_3

$$U_3 \quad = \quad b_3(x) \cdot \, D_x^3 \quad + b_2(x) \cdot \, D_x^2 \quad + b_1(x) \cdot \, D_x \quad + b_0(x), \qquad \text{(A.1)}$$

such that‡:

$$\tilde{\Omega}_4 \cdot U_3 \quad = \quad L_3 \cdot \, \Omega_4 \qquad \text{(A.2)}$$

We choose L_3 such that $\tilde{\Omega}_4$ is monic ($\tilde{\Omega}_4 \, = \, D_x^4 \, + \, \cdots$). Of course, we also have the (adjoint) relation:

$$adjoint(U_3) \cdot \, adjoint(\tilde{\Omega}_4) \quad = \quad adjoint(\Omega_4) \cdot \, adjoint(L_3). \qquad \text{(A.3)}$$

Furthermore, it is shown in Appendix B.1 that any operator satisfying the Calabi-Yau condition (94), is homomorphic to its adjoint, up to a conjugation by the square root of its Wronskian:

$$u(x) \cdot \, adjoint(\Omega_4) \quad = \quad \Omega_4 \cdot \, u(x). \qquad \text{(A.4)}$$

Combining (A.2), (A.3) and (A.4) one straightforwardly deduces:

$$\tilde{\Omega}_4 \cdot \, Y_6 \quad = \quad adjoint(Y_6) \cdot \, adjoint(\tilde{\Omega}_4), \qquad \text{(A.5)}$$

where the order-six operator Y_6 reads:

$$Y_6 \quad = \quad U_3 \cdot \, u(x) \cdot \, adjoint(L_3). \qquad \text{(A.6)}$$

†† DEtools in Maple.

‡ Given Ω_4 and U_3, in order to build the equivalent operator $\tilde{\Omega}_4$, just perform, in Maple, the LCLM of Ω_4 and U_3, and, then, right-divide by U_3.

Let us introduce the two operators N_2 and Z_2 corresponding to the euclidean division of Y_6 by $adjoint(\tilde{\Omega}_4)$:

$$Y_6 \quad = \quad N_2 \cdot adjoint(\tilde{\Omega}_4) \quad + \; Z_2. \qquad \text{(A.7)}$$

N_2 is of course an order-two operator, but, noticeably, Z_2 is *also* an *order-two* operator instead of an order-three operator one could expect generically.

Furthermore, and noticeably, N_2 is an order-two *self-adjoint* operator such that:

$$\frac{1}{b_3(x)} \cdot N_2 \cdot \frac{1}{b_3(x)} \quad = \quad u(x) \cdot \Bigl(D_x^2 \; - \frac{d\ln(1/u(x))}{dx} \cdot D_x\Bigr)$$
$$- \; u(x) \cdot \Bigl(\frac{db_2(x)}{dx} \; + b_2(x)^2 \; - 2\, b_1(x)\Bigr)$$
$$- \; u(x) \; - \; \frac{d^2u(x)}{dx^2} \; + \; \frac{2}{x} \cdot \Bigl(\frac{du(x)}{dx}\Bigr)^2. \qquad \text{(A.8)}$$

A consequence of the self-adjoint character of N_2 is that one also has the "adjoint" relation† of (A.7):

$$adjoint(Y_6) \quad = \quad \tilde{\Omega}_4 \cdot N_2 \quad + \; adjoint(Z_2). \qquad \text{(A.9)}$$

Combining (A.5), (A.7) and (A.9) one deduces the following homomorphisms of $\tilde{\Omega}_4$ with its adjoint, with an *order-two* intertwiner:

$$\tilde{\Omega}_4 \cdot Z_2 \;\; = \;\; adjoint(Z_2) \cdot adjoint(\tilde{\Omega}_4), \qquad \text{(A.10)}$$

Let us now perform the euclidean division of $adjoint(\tilde{\Omega}_4)$ by Z_2:

$$adjoint(\tilde{\Omega}_4) \quad = \quad A_2 \cdot Z_2 \quad + \; A_0 \qquad \text{(A.11)}$$

where A_2 is an order-two operator and, surprisingly, A_0 is not an order-one operator, *but a function* (order zero). Of course (and using the fact that the adjoint of two even order operators is the sum of the adjoints) one also has the "adjoint relation" of (A.11), namely

$$\tilde{\Omega}_4 \quad = \quad adjoint(Z_2) \cdot adjoint(A_2) \quad + \; A_0. \qquad \text{(A.12)}$$

In fact, and noticeably A_2 is a *self-adjoint operator*. Combining (A.10), (A.11) and (A.12), one immediately deduces that Z_2 is *conjugated to its adjoint*, or equivalently, that the following order-two operator Z_2^s *is self-adjoint*:

$$Z_2^s \quad = \quad A_0 \cdot Z_2 \quad = \quad adjoint(Z_2) \cdot A_0. \qquad \text{(A.13)}$$

One finds out that the order-four operator $\tilde{\Omega}_4$ can, in fact, be written in terms of a remarkable decomposition with two *order-two self-adjoint operators*:

$$\tilde{\Omega}_4 \quad = \quad Z_2^s \cdot \frac{1}{A_0} \cdot A_2 \quad + \quad A_0. \qquad \text{(A.14)}$$

One then deduces the homomorphisms of $\tilde{\Omega}_4$ with its adjoint:

$$A_2 \cdot \frac{1}{A_0} \cdot \tilde{\Omega}_4 \quad = \quad adjoint(\tilde{\Omega}_4) \cdot \frac{1}{A_0} \cdot A_2. \qquad \text{(A.15)}$$

to be compared with

$$\tilde{\Omega}_4 \cdot \frac{1}{A_0} \cdot Z_2^s \quad = \quad Z_2^s \cdot \frac{1}{A_0} \cdot adjoint(\tilde{\Omega}_4). \qquad \text{(A.16)}$$

† One uses the fact that the adjoint of the sum of an order-six and an order-two operator is the sum of these adjoints.

Note that one also has a relation similar to (A.5) but on Ω_4. If one introduces

$$Z_8 \quad = \quad adjoint(L_3) \cdot \frac{1}{A_0} \cdot A_2 \cdot U_3 \cdot u(x), \qquad \text{(A.17)}$$

one has

$$adjoint(Z_8) \cdot \Omega_4 \cdot u(x) \quad = \quad \Omega_4 \cdot u(x) \cdot Z_8 \qquad \text{(A.18)}$$

where $\Omega_4 \cdot u(x)$ also verifies the Calabi-Yau condition and is *actually self-adjoint*. Denoting $\Omega_4^{(s)} = \Omega_4 \cdot u(x)$ this gives

$$adjoint(Z_8) \cdot \Omega_4^{(s)} \quad = \quad \Omega_4^{(s)} \cdot Z_8. \qquad \text{(A.19)}$$

One discovers that

$$Z_8 \quad = \quad X_4 \cdot \Omega_4^{(s)} \; - 1. \qquad \text{(A.20)}$$

where X_4 is *actually self-adjoint*. One straightforwardly deduces

$$adjoint(X_4) \cdot adjoint(Z_8) \quad = \quad X_4 \cdot adjoint(Z_8) \quad = \quad Z_8 \cdot X_4. \qquad \text{(A.21)}$$

If one does not normalize the order-four operators $\tilde{\Omega}_4$ and Ω_4 to be monic, these results are easily modified, mutatis mutandis:

$$\Omega_4 \;\rightarrow\; \alpha(x) \cdot \Omega_4, \qquad \tilde{\Omega}_4 \;\rightarrow\; \beta(x) \cdot \tilde{\Omega}_4, \qquad L_3 \;\rightarrow\; \beta(x) \cdot L_3 \cdot \alpha(x)^{-1}$$
$$u(x) \;\rightarrow\; \alpha(x) \cdot u(x), \qquad Y_6 \;\rightarrow\; Y_6 \cdot \beta(x), \qquad Z_2 \;\rightarrow\; Z_2 \cdot \beta(x), \qquad \text{(A.22)}$$
$$SZ_2 \;\rightarrow\; \beta(x) \cdot SZ_2 \cdot \beta(x), \quad A_0 \;\rightarrow\; \beta(x) \cdot A_0, \quad (A_2, N_2, U_3) \;\rightarrow\; (A_2, N_2, U_3).$$

These calculations could have been performed, in another way, considering the other homomorphism relation between $\tilde{\Omega}_4$ and Ω_4, instead of (A.2):

$$V_3 \cdot \tilde{\Omega}_4 \quad = \quad \Omega_4 \cdot M_3, \qquad \text{(A.23)}$$

and its "adjoint" relation:

$$adjoint(\tilde{\Omega}_4) \cdot adjoint(V_3) \quad = \quad adjoint(M_3) \cdot adjoint(\Omega_4). \qquad \text{(A.24)}$$

Combining (A.23), (A.24) and (A.4), one easily deduces:

$$adjoint(\tilde{\Omega}_4) \cdot Z_6 \quad = \quad adjoint(Z_6) \cdot \tilde{\Omega}_4, \qquad \text{(A.25)}$$

where:

$$Z_6 \quad = \quad adjoint(V_3) \cdot \frac{1}{u(x)} \cdot M_3. \qquad \text{(A.26)}$$

The rightdivision of this order-six operator by $\tilde{\Omega}_4$ yields:

$$Z_6 \quad = \quad P_2 \cdot \tilde{\Omega}_4 \; + \; A_2 \cdot \frac{1}{A_0}, \qquad \text{(A.27)}$$

where P_2 is an order-two self-adjoint operator (much more involved than N_2 ...).

Appendix B. Calabi-Yau conditions

Appendix B.1. Calabi-Yau conditions and self-adjoint operators

Since most of the linear differential operators of order four with polynomial coefficients, we have encountered in lattice statistical mechanics [17, 11, 13, 52, 16, 20], enumerative combinatorics, are *globally nilpotent* [8], and thus their Wronskian are N-th root of *rational functions*, let us write, without any loss of generality, the coefficient $a_3(x)$ of operator Ω_4 (see (92)), in the log-derivative form:

$$a_3(x) \quad = \quad - \, {{ d \, \ln(w(x))} \over {dx}} \qquad \hbox{with:} \qquad w(x) \, = \, u(x)^2. \qquad \qquad (B.1)$$

Global nilpotence, even being Fuchsian, corresponds to the Wronskian $w(x) \, = \, u(x)^2$ being N-th root of a *rational function*.

It is straightforward to verify that if Ω_4 satisfies the Calabi-Yau condition (94), then:

$$u(x) \cdot adjoint(\Omega_4) \quad = \quad \Omega_4 \cdot u(x). \qquad \qquad (B.2)$$

In other words, the Calabi-Yau condition (94) necessarily means that the order-four operator (92) is, not only homomorphic to its adjoint, but conjugated to its adjoint. The following conjugate of Ω_4 is *self-adjoint*:

$$\tilde{\Omega}_4 \quad = \quad u(x)^{-1/2} \cdot \Omega_4 \cdot u(x)^{1/2}. \qquad \qquad (B.3)$$

This can be easily checked on all the ODEs of the large list of Calabi-Yau ODEs displayed in [5].

Conversely, let us impose that an order-four operator (92) is conjugated to its adjoint. Again it is straightforward to see that relation (B.3) yields:

$$a_1(x) \quad = \quad {{da_2(x)} \over {dx}} \, - {{a_2(x)} \over {u(x)}} \cdot {{du(x)} \over {dx}} \, + {{1} \over {u(x)}} \cdot {{d^3u(x)} \over {dx^3}} \qquad (B.4)$$
$$\quad + 6 \cdot \Bigl(\Bigl({{1} \over {u(x)}} \cdot {{du(x)} \over {dx}} \Bigr)^3 \, - {{1} \over {u(x)^2}} \cdot {{du(x)} \over {dx}} \cdot {{d^2u(x)} \over {dx^2}} \Bigr),$$

which is *nothing but the Calabi-Yau condition* (94) taking into account (B.1).

In other words, the *Calabi-Yau condition* (94) *is equivalent to say that an order-four operator is conjugated to its adjoint*. This result can also be found in Bogner (see [22]).

Appendix B.1.1. Strong Calabi-Yau conditions versus self-adjoint conditions on operators: higher order operators

An operator of order five is self-adjoint if it is of the form:

$$L_5 \quad = \quad a_5(x) \cdot D_x^5 \, + \, {{5} \over {2}} \cdot {{da_5(x)} \over {dx}} \cdot D_x^4 \, \, + a_3(x) \cdot D_x^3$$
$$\quad + \, \Bigl({{3} \over {2}} \, {{da_3(x)} \over {dx}} \, - {{5} \over {2}} \, {{d^3a_5(x)} \over {dx^3}} \Bigr) \cdot D_x^2 \, \, + a_1(x) \cdot D_x \qquad (B.5)$$
$$\quad + \, \Bigl({{1} \over {2}} \cdot {{da_1(x)} \over {dx}} \, + {{1} \over {2}} \cdot {{d^5a_5(x)} \over {dx^5}} \, - {{1} \over {4}} \cdot {{d^3a_3(x)} \over {dx^3}} \Bigr).$$

Its *symmetric square is of order fourteen* instead of the order fifteen one could expect generically. In other words this (exactly) self-adjoint operator, or an order-five operator conjugated of (B.5) by an arbitrary function, satisfies the symmetric Calabi-Yau condition (that its symmetric square is of order fourteen).

An operator of order six is self-adjoint if it is of the form:

$$L_6 \; = \; a_6(x) \cdot D_x^6 \; + 3 \cdot \frac{da_6(x)}{dx} \cdot D_x^5 \; + a_4(x) \cdot D_x^4 + \Bigl(2 \cdot \frac{da_4(x)}{dx} - 5 \cdot \frac{d^3a_6(x)}{dx^3}\Bigr) \cdot D_x^3 \; + a_2(x) \cdot D_x^2 + \Bigl(\frac{da_2(x)}{dx} - \frac{d^3a_4(x)}{dx^3} + 3 \cdot \frac{d^5a_6(x)}{dx^5}\Bigr) \cdot D_x \; + a_0(x). \tag{B.6}$$

Its *exterior square is of order fourteen* instead of the order fifteen one could expect generically. In other words this (exactly) self-adjoint operator, or an order-six operator conjugated of (B.6) by an arbitrary function, satisfies the Calabi-Yau condition (that its exterior square is of order fourteen), generalization to order six of the order-four Calabi-Yau condition (94).

It is straightforward to verify that an operator conjugated of a self-adjoint operator of order N verifies, for *any* even order N, the generalization to order N of the order-four Calabi-Yau condition (94) and for *any* odd order N, the generalization to order N of the order-three *symmetric Calabi-Yau condition* (96).

Of course the reciprocal, which is true for order-three and four operators (see (97) and (B.4)), is not true for higher orders. For instance, let us introduce the order-five operator M_5 non-trivially homomorphic to the self-adjoint operator (B.5):

$$M_5 \cdot D_x \; = \; \frac{1}{a_5(x)} \cdot \Bigl(D_x - \frac{1}{W(x)} \cdot \frac{dW(x)}{dx}\Bigr) \cdot L_5, \tag{B.7}$$

where: $$W(x) \; = \; 2\,\frac{da_1(x)}{dx} - \frac{d^3a_3(x)}{dx^3} + 2\,\frac{d^5a_5(x)}{dx^5},$$

This operator also verifies the order-five *symmetric Calabi-Yau condition* (96): its symmetric square is also of order fourteen. This result generalizes with M_5

$$M_5 \cdot (D_x + \rho(x)) \; = \; \frac{1}{a_5(x)} \cdot \Bigl(D_x - z(x)\Bigr) \cdot L_5, \tag{B.8}$$

where $z(x)$ is a quite involved rational expression of $a_1(x)$, $a_3(x)$, $a_5(x)$, $\rho(x)$ and their derivatives.

Similarly the order-six operator M_6 equivalent of the self-adjoint operator (B.6):

$$M_6 \cdot D_x \; = \; \frac{1}{a_6(x)} \cdot \Bigl(D_x - \frac{1}{a_0(x)} \cdot \frac{da_0(x)}{dx}\Bigr) \cdot L_6,$$

verifies the order-six *Calabi-Yau condition* (94): its exterior square is also of order fourteen.

This result generalizes with M_6 given by

$$M_6 \cdot (D_x + \rho(x)) \; = \; \frac{1}{a_6(x)} \cdot \Bigl(D_x - z(x)\Bigr) \cdot L_6, \tag{B.9}$$

where $z(x)$ is a quite involved rational expression of $a_0(x)$, $a_2(x)$, $a_4(x)$, $a_6(x)$, $\rho(x)$ and their derivatives.

The order-seven operator M_7 equivalent of the order-seven self-adjoint operator L_7:

$$M_7 \cdot (D_x^2 + \rho_1(x) \cdot D_x + \rho_2(x)) \; = \; \frac{1}{a_7(x)} \cdot (D_x^2 + c_1(x) \cdot D_x + c_2(x)) \cdot L_7,$$

verifies the order-seven *Calabi-Yau condition* that its symmetric square is of order 27 (instead of the generic order 28).

Similarly the order-eight operator M_8 equivalent of the order-seven self-adjoint operator L_8:

$$M_8 \cdot (D_x^2 + \rho_1(x) \cdot D_x + \rho_2(x)) \;=\; \frac{1}{a_7(x)} \cdot (D_x^2 + c_1(x) \cdot D_x + c_2(x)) \cdot L_8,$$

verifies the order-seven *symmetric Calabi-Yau condition* that its symmetric square is of order 27 (instead of the generic order 28). These last results can easily be generalized. For instance for the order-nine and order-ten self-adjoint operators L_9, L_{10} the corresponding equivalent operators M_9, M_{10} obtained from the LCLM of L_9 or L_{10} with an *order-three* operator verify respectively the order-nine *symmetric Calabi-Yau condition* (namely the symmetric square of M_9 is of order 44 instead of 45) and the order-ten Calabi-Yau condition (namely that the symmetric and exterior squares of M_9 and M_{10} are of order 44 instead of 45), and so on ...

Appendix B.2. Equivalence of operators satisfying the Calabi-Yau conditions

Let us recall some examples (see equation (O.37) in [19]) of operators satisfying the Calabi-Yau conditions (94). The order-four (μ-dependent) linear differential operator

$$\mathcal{C}(\mu) \;=\; 16 \cdot \theta^2 \cdot (\theta - 1)^2 \qquad \text{(B.10)}$$
$$- \; x \cdot (2\theta + 1 - \mu) \cdot (2\theta + 1 + \mu) \cdot (2\theta - 1 - \mu) \cdot (2\theta - 1 + \mu),$$

is such that *its exterior square is actually of order five.* These operators are *irreducible* for even integer values of μ, but factor in a product of order-two and two order-one operators for odd integer values of μ. Operator (B.10) has simple hypergeometric solutions for *any value* of μ:

$$x \cdot {}_4F_3\Big([\frac{\mu+3}{2}, \frac{-\mu+3}{2}, \frac{\mu+1}{2}, \frac{-\mu+1}{2}], [1, 2, 2]; \; x). \qquad \text{(B.11)}$$

These operators (B.10) are, for different *even integer values* of μ, non-trivially homomorphic (and similarly, these operators (B.10) are, non-trivially homomorphic for different odd integer values of μ). One of the simplest examples of homomorphism reads:

$$\mathcal{C}(0) \cdot U_3 \;=\; V_3 \cdot \mathcal{C}(2), \qquad \text{with:}$$
$$U_3 \;=\; (32\,\theta^3 - 80\,\theta^2 + 72\,\theta - 27) \; - 4 \cdot x \cdot (2\,\theta + 1)\,(2\,\theta - 1)\,(2\,\theta - 3),$$
$$V_3 \;=\; (32\,\theta^3 - 80\,\theta^2 + 72\,\theta - 27) \; - 4 \cdot x \cdot (2\,\theta + 1)\,(2\,\theta - 1)^2.$$

More generally, for $\mu = 2\,N$ one has

$$\mathcal{C}(0) \cdot U_3 \;=\; V_3 \cdot \mathcal{C}(2\,N) \qquad \text{(B.12)}$$

in the order-three operators U_3 and V_3, the degree in x being N. We have an *infinite number of equivalent operators* satisfying the (strong) Calabi-Yau condition. This is a consequence of the following homomorphism between $\mathcal{C}(\mu)$ and $\mathcal{C}(\mu+2)$ valid for *any value of* μ (μ being not even a rational number):

$$\mathcal{C}(\mu) \cdot X_3 \;=\; Y_3 \cdot \mathcal{C}(\mu+2), \qquad \text{(B.13)}$$

where X_3 reads

$$4\,x \cdot \Big(2\,\big(3\,\mu^2 + 6\,\mu - 1\big) \cdot \theta + 4\,\mu^3 + 15\,\mu^2 + 14\,\mu - 1\Big)\,(2\,\theta - \mu - 1)\,(2\,\theta - \mu - 3)$$
$$- 32\,\big(3\,\mu^2 + 6\,\mu - 1\big) \cdot \theta^3 + 16\,\big(2\,\mu^3 + 15\,\mu^2 + 20\,\mu - 5\big) \cdot \theta^2$$
$$- 8\,\big(\mu^2 + 2\,\mu - 1\big)\,(\mu+3)^2 \cdot \theta + (\mu+3)^3\,(\mu^2 - 1), \qquad \text{(B.14)}$$

and Y_3 reads:

$$4\,x \cdot \Bigl(2\,\bigl(3\,\mu^2+6\,\mu-1\bigr)\cdot\,\theta\,+4\,\mu^3+9\,\mu^2+2\,\mu+1\Bigr)\,\,(2\,\theta-\mu+1)\,(2\,\theta-\mu-1)$$
$$-\,32\,\bigl(3\,\mu^2+6\,\mu-1\bigr)\cdot\,\theta^3\,+16\,\bigl(2\,\mu^3+15\,\mu^2+20\,\mu-5\bigr)\cdot\,\theta^2$$
$$-\,8\,\bigl(\mu^2+2\,\mu-1\bigr)\,\,(\mu+3)^2\,\cdot\,\theta\,\,+\,(\mu+3)^3\,(\mu^2-1), \qquad\qquad (B.15)$$

the two order-three operators being themselves homomorphic†.

To sum-up: This shows that a solution of the reduction (by equivalence of operators) of operators satisfying the weak Calabi-Yau condition to operators satisfying the (strong) Calabi-Yau condition is *not unique* since one can find, for some examples, an *infinite number* of homomorphic irreducible operators satisfying the (strong) Calabi-Yau condition.

Appendix B.2.1. Decomposition

Let us consider, for instance, the order-four operator obtained by the rightdivision by θ of the LCLM of $\mathcal{C}(\mu)$ and θ. This operator $\mathcal{M}(\mu)$ satisfies the weak Calabi-Yau condition: its exterior square has a rational solution $1/(x-1)$ *independent of* μ, and homomorphic to the infinite number of operators $\mathcal{C}(\mu+2\,N)$ (N is any integer), satisfying the (strong) Calabi-Yau condition that their exterior square is of order five.

One has a decomposition similar to (100), namely:

$$\mathcal{M}(\mu) \,\,=\,\, L_2\cdot\,a(x)\cdot\,M_2\,\,+\,\,\frac{(\mu^2-1)^2}{16\,a(x)}, \qquad\qquad a(x)\,\,=\,\,x^3\cdot\,(x-1),$$
$$L_2 \,\,=\,\, \frac{1}{(x-1)^4\,\,x^5}\cdot\,\Bigl(x\cdot\,(\theta\,-4)\,\,-(\theta\,-2)\Bigr)\cdot\,\Bigl(x\cdot\,(\theta\,-3)\,\,-(\theta\,-2)\Bigr),$$
$$M_2 \,\,=\,\, \frac{1}{x^2}\cdot\,\Bigl(x\cdot\,(\theta^2\,-\,\frac{\mu^2\,+1}{2})\,\,-\theta\cdot\,(\theta-1)\Bigr), \qquad\qquad (B.16)$$

where L_2 and M_2 are self-adjoint operators.

Let us consider, instead of $\mathcal{C}(\mu)$, the self-adjoint operator $C_s(\mu)\,=\,1/x^2\cdot\mathcal{C}(\mu)$, which satisfies the Calabi-Yau condition and instead of $\mathcal{M}(\mu)$ the operator $M(\mu)\,=\,a(x)\cdot\mathcal{M}(\mu)$:

$$M(\mu) \,\,=\,\, -16\cdot\,a(x)\cdot\,L_2\cdot\,a(x)\cdot\,M_2\,\,-\,\,(\mu^2-1)^2. \qquad\qquad (B.17)$$

One has the following homomorphism between $C_s(\mu)$ and the adjoint of $M(\mu)$:

$$(\theta\,+1)\cdot\,adjoint(M(\mu)) \,\,=\,\, C_s(\mu)\cdot\,x\cdot\,(\theta\,+1). \qquad\qquad (B.18)$$

One notes that the exterior square of $adjoint(M(\mu))$, as well as the exterior square of $(\theta\,+1)\cdot adjoint(M(\mu))$, have the rational solution $1/x^3$. This is in agreement with the fact that the exterior square of $adjoint(\mathcal{M}(\mu))$ is nothing but the Wronskian of the self-adjoint operator L_2, namely, $x^3\cdot(x-1)^2\,=\,a(x)^2/x^3$. One thus sees that, if the exterior square of $C_s(\mu)$ has no rational solution, $C_s(\mu)\cdot x\cdot(\theta+1)$ has a rational solution, namely $1/x^3$.

Appendix B.3. Calabi-Yau conditions preserved by the formal adjoint

Let us consider a (monic) order-four operator with a rational (resp. N-th root of rational) Wronskian $W(x)$:

$$\Omega_4 \,\,=\,\, D_x^4\,\,-\,\,\frac{d\ln(W(x))}{dx}\cdot\,D_x^3\,\,+\,a_2(x)\cdot\,D_x^2\,\,+\,a_1(x)\cdot\,D_x\,\,+\,a_0(x).$$

† We have the following "tower" of homomorphisms: $X_3\cdot X_2\,=\,Y_2\cdot Y_3$, $X_2\cdot X_1\,=\,Y_1\cdot Y_2$, $X_1\cdot(x-1)\,=\,(x-1)\cdot Y_1$.

One has the following *conjugation relation between the exterior square of this operator and the exterior square of its adjoint*:

$$W(x) \cdot Ext^2(adjoint(\Omega_4)) \quad = \quad Ext^2(\Omega_4) \cdot W(x). \tag{B.19}$$

From this conjugation relation it is straightforward to deduce that if Ω_4 satisfies the *Calabi-Yau condition* (94), i.e. its exterior square is of order five instead of six, the exterior square of its adjoint will also be of order five: the Calabi-Yau condition (94) is thus *preserved by the adjoint transformation*. From this conjugation relation (B.19) it is also straightforward to deduce that if the operator satisfies the *weak Calabi-Yau condition*, i.e. its exterior square has a rational (resp. N-th root of rational) solution, this will also be the case for its adjoint: the exterior square of the adjoint of this operator will also have a rational solution, which is nothing but the previous rational (resp. N-th root of rational) solution divided by $W(x)$ the Wronskian of the operator. In other words, the *weak Calabi-Yau condition is preserved by the adjoint*.

Appendix B.3.1. Decomposition of order-four operators

Let us consider an operator M_4 of the form (100) where L_2 and M_2 are two (general) self-adjoint operators (101) and (102). Note that $a(x) \cdot M_2 \cdot a(x)$ is also a self-adjoint operator, so up to an overall factor $\rho(x)$, one can consider, without any restriction, the form

$$M_4 \quad = \quad M_2 \cdot L_2 \quad + \quad \lambda, \tag{B.20}$$

where L_2 and M_2 are two (general) self-adjoint operators (see (101) and (102)).

The exterior square of an order-four operator of the form (B.20) (up to an overall factor $\rho(x)$) has the rational solution $1/\alpha_2(x)$. Switching to the adjoint amounts to permuting the two self-adjoint operators L_2 and M_2. The order-four operator M_4 is not monic. Denoting $M_4^{(u)}$ the order-four operator M_4 in a monic form we have $M_4 = \rho(x) \cdot M_4^{(u)}$ with $\rho(x) = \alpha_2(x) \cdot \beta_2(x)$. Thus one has the following relations† between their adjoints:

$$\begin{aligned} adjoint(M_4) \; &= \; adjoint(M_4^{(u)}) \cdot \rho(x), \\ Ext^2(adjoint(M_4)) \; &= \; Ext^2(adjoint(M_4^{(u)}) \cdot \rho(x)) \\ &= \; \frac{1}{\rho(x)^2} \cdot Ext^2(adjoint(M_4^{(u)})) \cdot \rho(x)^2. \end{aligned} \tag{B.21}$$

One has the previous relation (B.19) between the exterior square of the *monic* operator, its Wronskian, and the adjoint of the *monic* operator:

$$W(x) \cdot Ext^2(adjoint(M_4^{(u)})) \quad = \quad Ext^2(M_4^{(u)}) \cdot W(x), \qquad \text{where:}$$

$$W(x) \quad = \quad \frac{\beta_2(x)}{\rho(x)^2 \cdot \alpha_2(x)} \quad = \quad \frac{1}{\beta_2(x) \cdot \alpha_2(x)^3}, \tag{B.22}$$

which can be rewritten on the exterior square of M_4 and its adjoint:

$$\frac{\beta_2(x)}{\alpha_2(x)} \cdot Ext^2(adjoint(M_4)) \quad = \quad Ext^2(M_4) \cdot \frac{\beta_2(x)}{\alpha_2(x)}. \tag{B.23}$$

Since the exterior square of M_4 of the form (B.20) has the rational solution $1/\alpha_2(x)$, the last relation (B.23) is compatible with the fact that the exterior square of the adjoint of M_4 (of the form (B.20)) has the rational solution $1/\beta_2(x)$.

† Since by definition the exterior square of an operator is normalized to be a *monic* operator.

Remark: Let us consider an operator C_4 with head coefficient $A_4(x)$ and Wronskian $W(x)$, satisfying the (strong) Calabi-Yau condition that its exterior square is of order five. If one has an intertwining relation

$$M_4 \cdot \tilde{R}(x) \cdot \Bigl(D_x - {{d \ln(R(x))} \over {dx}}\Bigr) \quad = \quad \tilde{S}(x) \cdot \Bigl(D_x - {{d \ln(S(x))} \over {dx}}\Bigr) \cdot C_4,$$

one has the simple relations:

$$\tilde{S}(x) \quad = \quad {{\beta_2(x) \cdot a(x)^2 \cdot \alpha_2(x) \cdot \tilde{R}(x)} \over {A_4(x)}},$$

$$W(x) \quad = \quad {{A_4(x) \cdot R(x)} \over {\tilde{R}(x)^4 \cdot \beta_2(x) \cdot \alpha_2(x)^3 \cdot a(x)^2 \cdot S(x)}}. \qquad (B.24)$$

Therefore there are some simple relations between $\alpha_2(x)$ and $\beta_2(x)$ that will correspond to the rational solutions of the exterior square of M_4, or of the adjoint of M_4.

Appendix B.4. Rational solutions for the exterior square of operators satisfying the weak Calabi-Yau condition

The fact that an order-four operator satisfying the weak Calabi-Yau condition, namely having a decomposition of the form (100), is such that its exterior square has a rational solution which is the inverse of the head coefficient of the right most order-two self-adjoint operator is, in fact, the consequence of an identity on the difference of two exterior squares.

A non-trivial identity exists between the difference of the following two exterior squares:

$$Ext^2(L_2 \cdot M_2 \, + \, \lambda) \, - \, Ext^2(L_2 \cdot M_2) \quad = \qquad (B.25)$$

$$- \, 4 \cdot {{\lambda} \over {\alpha_2(x) \, \beta_2(x)}} \cdot \Bigl(D_x \, - {{1} \over {2}} \cdot {{d \ln(\rho_L(x))} \over {dx}}\Bigr) \cdot \Bigl(D_x \, - {{1} \over {2}} \cdot {{d \ln(\rho_R(x))} \over {dx}}\Bigr), \qquad \hbox{where:}$$

$$\rho_L(x) \,\, = \,\, {{\mathcal{P}^2} \over {\alpha_2(x)^2 \cdot \beta_2(x)^2 \cdot w_L(x)^5 \cdot w_M(x)^5}}, \quad \rho_R(x) \,\, = \,\, {{w_L(x) \cdot w_M(x) \cdot \alpha_2(x)} \over {\beta_2(x)}},$$

where $\mathcal{P}$ is a slightly involved‡ polynomial expression, on exterior square of product of the two (not necessarily self-adjoint) operators

$$L_2 \quad = \quad \alpha_2(x) \cdot (D_x^2 \, - \, {{d \, \ln(w_L(x)} \over {dx}} \cdot D_x) \, + \, \alpha_0(x), \qquad (B.26)$$

$$M_2 \quad = \quad \beta_2(x) \cdot (D_x^2 \, - \, {{d \, \ln(w_M(x)} \over {dx}} \cdot D_x) \, + \, \beta_0(x). \qquad (B.27)$$

For self-adjoint operators like (101) and the decomposition (100), one has $\alpha_2(x) = a(x)/w_L(x)$ and $\beta_2(x) = a(x)/w_M(x)$, $\rho_R = w_M(x)^2$. Furthermore it is simple to see that the exterior square of the product $L_2 \cdot M_2$ has the Wronskian of M_2 as a solution. Therefore identity (B.25) means that the exterior square of $L_2 \cdot M_2 \, + \, \lambda$ has $w_M(x)$ the Wronskian of M_2 as a solution.

‡ Quadratic in $\alpha_2(x)$ and $\beta_2(x)$, linear in $\alpha_0(x)$ and $\beta_0(x)$, and polynomial in $w_L(x)$ and $w_M(x)$ and their derivatives up to third derivative.

Appendix C. Analysis of the order-seven operators associated with the exceptional Galois group $G_2(C)$

Appendix C.1. Solution-series of the order-seven operators

The solution-series $y_0^{(n)}$, analytic at $x = 0$, of the order-seven operators E_n, $n = 1, 2 \cdots$, given in section (5), are actually series with *integer coefficients* and read respectively

$$\begin{aligned}
y_0^{(1)} \; &= \; 1 \; + 2688\,x \, + 19707264\,x^2 \, + 191647334400\,x^3 \, + 2133255623587200\,x^4 \\
& \qquad + 25707449648409919488\,x^5 \; + \; \cdots \\
y_0^{(2)} \; &= \; 1 \; + 384\,x \, + 537984\,x^2 \, + 1097318400\,x^3 \, + 2680866518400\,x^4 \\
& \qquad + 7283382738960384\,x^5 \; + \; \cdots \\
y_0^{(3)} \; &= \; 1 \; + 1512\,x \, + 9885240\,x^2 \, + 95782780800\,x^3 \, + 1117658718099000\,x^4 \\
& \qquad + 14536396497887776512\,x^5 \; + \; \cdots \\
y_0^{(4)} \; &= \; 1 \; + 14976\,x \, + 1254798720\,x^2 \, + 159551671910400\,x^3 \\
& \quad + 24603126146687088000\,x^4 + 4241337041632715022974976\,x^5 \; + \; \cdots \\
y_0^{(5)} \; &= \; 1 \; + 2678400\,x \, + 65172299068800\,x^2 \, + 2494516941707677286400\,x^3 \\
& \qquad + 116986156694543894801624380800\,x^4 \\
& \qquad\quad + 6160069364202852097613676563979878400\,x^5 \; + \; \cdots
\end{aligned} \tag{C.1}$$

These order-seven linear differential operators are *globally nilpotent* [8, 40], the Jordan reduction of their p-curvature [40, 8] reading:

$$J_7 \; = \; \left[\begin{array}{ccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right], \tag{C.2}$$

of characteristic polynomial equal to the minimal polynomial $P(\lambda) \, = \, \lambda^7$.

These operators are MUM, so they have the traditional "triangular log structure" [20], the formal series solutions with a log being of the form $y_1^{(n)} \, = \, y_0^{(n)} \cdot \ln(x) + \tilde{y}_1^{(n)}$, $y_2^{(n)} \, = \, y_0^{(n)} \cdot \ln(x)^2/2 + \tilde{y}_1^{(n)} \cdot \ln(x) + \tilde{y}_2^{(n)}$, etc. The corresponding nomes (called "special coordinates" in [29]) are defined as $q^{(n)} \, = \, \exp(y_1^{(n)}/y_0^{(n)}) \, = \, x \cdot \exp(\tilde{y}_1^{(n)}/y_0^{(n)})$. These nomes correspond to series with *integer coefficients*, and read respectively:

$$\begin{aligned}
q^{(1)} \; &= \; x \; + 7040\,x^2 \, + 67555904\,x^3 \, + 747082784768\,x^4 \\
& \qquad + 8968272297124128\,x^5 \, + \; \cdots, \\
q^{(2)} \; &= \; x \; + 1152\,x^2 \, + 2150976\,x^3 \, + 4983447552\,x^4 \\
& \qquad + 13054714896672\,x^5 \, + \; \cdots,
\end{aligned}$$

$$q^{(3)} \;=\; x \;+5562\,x^2 \;+49552317\,x^3 \;+547802062578\,x^4 \;+6855142017357054\,x^5 \;+\;\cdots,$$

$$q^{(4)} \;=\; x \;+72576\,x^2 \;+8462979648\,x^3 \;+1230038144557056\,x^4 \;+203018472128017391904\,x^5 \;+\;\cdots,$$

$$q^{(5)} \;=\; x \;+20200320\,x^2 \;+689499895026240\,x^3 \;+29916247864887732510720\,x^4 \;+1488739080271271648779215102240\,x^5 \;+\;\cdots \tag{C.3}$$

Appendix C.1.1. Yukawa couplings

The Yukawa couplings

$$K(q) \;=\; \Bigl(q \cdot \frac{d}{dq}\Bigr)^2 \Bigl(\frac{y_2}{y_0}\Bigr), \tag{C.4}$$

of the five order-seven linear differential operators of section (5) are series with integer coefficients. Their expansion read respectively:

$$K^{(1)} \;=\; 1 \;+768\,q \;-2188032\,q^2 \;+2883403776\,q^3 \;-1360234636032\,q^4 \;-3787008084959232\,q^5 \;+\;\cdots$$

$$K^{(2)} \;=\; 1 \;+256\,q + 728320\,q^2 \;+1640611840\,q^3 \;+3618799525120\,q^4 \;+8043817914720256\,q^5 \;+\;\cdots$$

$$K^{(3)} \;=\; 1 \;+1485\,q \;+19708515\,q^2 \;+206970715890\,q^3 \;+2188620549305955\,q^4 \;+23409935555891063985\,q^5 \;+\;\cdots$$

$$K^{(4)} \;=\; 1 \;+29440\,q \;+4438662400\,q^2 \;+621410936504320\,q^3 \;+88605364227964837120\,q^4 \;+12835248124604913684029440\,q^5 \;+\;\cdots$$

$$K^{(5)} \;=\; 1 \;+17342208\,q + 687629971954944\,q^2 \;+30848876097264182771712\,q^3 \;+1428770297588004620323742981376\,q^4 \;+67528440221394152640448454407310942208\,q^5 \;+\;\cdots$$

These Yukawa couplings, in the x variable, read respectively:

$$K^{(1)} \;=\; 1 \;+768\,x \;+3218688\,x^2 \;+23958847488\,x^3 \;+229225505561856\,x^4 \;+2508123114368335872\,x^5 \;+\;\cdots$$

$$K^{(2)} \;=\; 1 \;+256\,x \;+1023232\,x^2 + 3869310976\,x^3 \;+14664270683392\,x^4 \;+56048323595665408\,x^5 \;+\;\cdots$$

$$K^{(3)} \;=\; 1 \;+1485\,x \;+27968085\,x^2 + 499793427495\,x^3 \;+9018524688844995\,x^4 \;+164714785807791646845\,x^5 \;+\;\cdots$$

$$K^{(4)} \;=\; 1 \;+29440\,x \;+6575299840\,x^2 + 1514841782026240\,x^3 \;+358624525635384843520\,x^4 \;+86502979031531419474001920\,x^5 \;+\;\cdots$$

$$K^{(5)} \; = \; 1 \; + 17342208\, x \, + 1037948123061504\, x^2 \\ + 70587017642949191073792\, x^3 \\ + 5045886607522553002548393221376\, x^4 \\ + 370665145887525483931062348265527902208\, x^5 \; + \; \cdots$$

Recalling our results in [18, 19], one can, for *higher order*-operators, define several Yukawa couplings from Wronskian-like determinants of the solutions, instead of a only one (C.4) for order-four operators:

$$K_3 \, = \, \frac{W_1^3 \cdot W_3}{W_2^3}, \qquad K_4 \, = \, \frac{W_1^8 \cdot W_4}{W_2^6}, \qquad K_5 \, = \, \frac{W_1^{15} \cdot W_5}{W_2^{10}}, \\ K_6 \, = \, \frac{W_1^{24} \cdot W_6}{W_2^{15}}, \qquad K_7 \, = \, \frac{W_1^{35} \cdot W_7}{W_2^{21}}. \qquad \text{(C.5)}$$

The well-known Yukawa coupling (C.4) is denoted K_3 in the previous set of "higher orders" Yukawa couplings (C.5) (see Appendix C.1 in [18]).

One remarks the following non-trivial identities for the five order-seven operators E_i:

$$K_4^{(i)} \, = \, (K_3^{(i)})^2, \qquad K_5^{(i)} \, = \, (K_3^{(i)})^3, \qquad K_6^{(i)} \, = \, (K_3^{(i)})^5, \\ K_7^{(i)} \, = \, (K_3^{(i)})^7, \qquad i \, = \, 1, \cdots, 5. \qquad \text{(C.6)}$$

Therefore, for the five E_i's, these various invariants *just reduce to the unique Yukawa coupling* K_3. These relations correspond respectively to the identities on the Wronskian-like determinants W_n:

$$W_1^2 \cdot W_4 \, = \, W_3^2, \qquad W_1^6 \cdot W_5 \, = \, W_2 \cdot W_3^3, \qquad W_1^9 \cdot W_6 \, = \, W_3^5, \\ W_1^{14} \cdot W_7 \, = \, W_3^7.$$

It had been seen (see (C.17) in [18, 19]), for order-four operators, that being self-adjoint up to a conjugation which yields that the Yukawa coupling K_3 is equal to the Yukawa coupling for the adjoint K_3^*, is nothing but relation $K_4 \, = \, K_3^2$, namely $W_1^2 \cdot W_4 \, = \, W_3^2$.

These relations are no longer valid for non self-adjoint (up to conjugation) order-seven operators. For order-seven operators taking the adjoint operator amounts to performing the following involutive transformation on the W_n's ($W_0 \, = \, 1$):

$$W_n \quad \longleftrightarrow \quad W_n^* \, = \, \frac{W_{7-n}}{W_7}, \qquad W_n \quad \longleftrightarrow \quad W_7^* \, = \, \frac{1}{W_7}, \qquad \text{(C.7)}$$

and, consequently, the Yukawa couplings (C.5) for the adjoint operator read:

$$K_3^* \, = \, \frac{W_6^3 \cdot W_4}{W_7 \cdot W_5^3} \, = \, \frac{K_6^3 \cdot K_4}{K_5^3 \cdot K_7}, \qquad K_4^* \, = \, \frac{W_6^8 \cdot W_3}{W_7^3 \cdot W_5^6} \, = \, \frac{K_6^8 \cdot K_3}{K_5^6 \cdot K_7^3}, \\ K_5^* \, = \, \frac{W_6^{15} \cdot W_2}{W_7^6 \cdot W_5^{10}} \, = \, \frac{K_6^{15}}{K_5^{10} \cdot K_7^6}, \qquad K_6^* \, = \, \frac{W_6^{24} \cdot W_1}{W_7^{10} \cdot W_5^{15}} \, = \, \frac{K_6^{24}}{K_5^{15} \cdot K_7^{10}}, \\ K_7^* \, = \, \frac{W_6^{35}}{W_7^{15} \cdot W_5^{21}} \, = \, \frac{1}{K_7}. \qquad \text{(C.8)}$$

Note, for the order-seven operator $\hat{E}_1^{(1)}$ (obtained from $\hat{E}_1$ by taking the LCLM with D_x and rightdividing by D_x), the Yukawa couplings (C.5), as well as these adjoint Yukawa couplings (C.8), as well as the W_n's, as well as the W_n's of the adjoint operators, are still globally bounded (see [18, 19]).

Do note that the relations (C.6) *are such that* $K_3^* = K_3$, $K_4^* = K_4$, $K_5^* = K_5$, $K_6^* = K_6$.

Taking into account the transformations of the W_n's by a pullback $p(x)$ one finds (with $v = p'$):

$$(W_1, W_2, W_3, W_4, W_5, W_6, W_7) \longrightarrow (W_1, v \cdot W_2, v^3 \cdot W_3, v^6 \cdot W_4, v^{10} \cdot W_5, v^{15} \cdot W_6, v^{21} \cdot W_7), \tag{C.9}$$

together with the obvious homogeneity

$$(W_1, W_2, W_3, W_4, W_5, W_6, W_7) \longrightarrow (u \cdot W_1, u^2 \cdot W_2, u^3 \cdot W_3, u^4 \cdot W_4, u^5 \cdot W_5, u^6 \cdot W_6, u^7 \cdot W_7), \tag{C.10}$$

one finds, if one seeks for invariants of the form $W_1^a \cdot W_n / W_2^b / W_3^c$ that the only invariants compatible with these symmetries are the (C.5) together with†

$$\frac{K_4}{K_3^2}, \qquad \frac{K_5}{K_3^3}, \qquad \frac{K_6}{K_3^5}, \qquad \frac{K_7}{K_3^7}. \tag{C.11}$$

Appendix D. Exceptional Galois groups: two and three parameter operators

Appendix D.1. Exceptional Galois groups: two-parameter deformation of E_1

Let us introduce the following two-parameter deformation of E_1 (it amounts to changing $F(x) \rightarrow F(x) \cdot (x/(1-x))^{1/2}$ in the P_2 of [30] then changing $x \rightarrow (x-1)/x$):

$$\begin{aligned}\Omega(p, q) \,\, = \,\, & \theta \cdot (\theta^2 - p^2) \cdot (\theta^2 - q^2) \cdot (\theta^2 - (p+q)^2) \\ & - 128 \cdot x \cdot \Big((48\,\theta^4 + 96\,\theta^3 + 124\,\theta^2 + 76\,\theta + 21)\,(2\,\theta + 1)^3 + 16\,\Sigma^2 \cdot (2\,\theta + 1)^2 \\ & \qquad - 8\,\Sigma \cdot (8\,\theta^2 + 8\,\theta + 5) \cdot (2\,\theta + 1)^2 - 64\,p^3\,q^3 \Big) \\ & + 4194304 \cdot x^2 \cdot (\theta + 1) \cdot \Big(12\,(\theta + 1)^2 + 11 - 8\,\Sigma \Big) \cdot (2\,\theta + 1)^2 \cdot (2\,\theta + 3)^2 \\ & - 34359738368 \cdot x^3 \cdot (2\,\theta + 5)^2 \cdot (2\,\theta + 1)^2 \cdot (2\,\theta + 3)^3, \end{aligned} \tag{D.1}$$

where $\Sigma = p^2 + p\,q + q^2$. In the $p = q = 0$ limit the previous two-parameter order-seven operator reduces to $\hat{E}_1$.

For *arbitrary values of* p and q the operator $x^{-1} \cdot \Omega(p, q)$ is *actually self-adjoint.* Note that $\Omega(p, 0)$ and $\Omega(p+1, 0)$ are homomorphic with an order-six intertwiner for *any value of* p. Also note that $\Omega(p, 1/2)$ and $\Omega(p+r, 1/2)$ are homomorphic with an order-six intertwiner for *any value of* p, for $r = 1/2, 1/3, 1/6, 1/11$.

† These ratio being equal to 1 in our case.

Appendix D.2. Exceptional Galois groups: families with three parameters

Let us consider the following order-seven operator† depending on *three* parameters a, c, d, (here σ denotes $b^2 + b\,c + c^2$):

$$\begin{aligned}\Omega_{a,b,c} \;=\;\; & \theta \cdot (\theta^2 - b^2) \cdot (\theta^2 - c^2) \cdot (\theta^2 - (b+c)^2) \\ & - x \cdot (2\,\theta + 1) \cdot (\theta + a) \cdot (\theta + 1 - a) \cdot \Bigl(\theta \cdot (\theta + 1) \cdot (\theta^2 + \theta + 1 - \sigma) \\ & \quad + 2\,a \cdot (1 - a) \cdot (\theta^2 + \theta + 1 - \sigma - a \cdot (1 - a))\Bigr) \\ & + x^2 \cdot (\theta + 1) \cdot (\theta + a) \cdot (\theta + 1 - a) \cdot (\theta + a + 1) \cdot (\theta + (1 - a) + 1) \\ & \qquad \times (\theta + 2\,a)\,(\theta + 2 \cdot (1 - a)). \end{aligned} \qquad \text{(D.2)}$$

On this explicit expression one sees obviously that (D.2) is (b, c)-symmetric, $\Omega_{a,b,c} = \Omega_{a,c,b}$ and that it is invariant by the $a \leftrightarrow 1 - a$ involution, $\Omega_{a,b,c} = \Omega_{1-a,b,c}$. Less obviously one notes that $\Omega_{a,b,c}$ and $\Omega_{a+N,b+M,c+P}$ are homomorphic *for any value of the three integers* N, M, P. This operator can easily be turned into a *self-adjoint* operator $\Omega^s_{a,b,c} = x^{-1/2} \cdot \Omega_{a,b,c} \cdot x^{1/2}$ (or the self-adjoint operator $x^{-1} \cdot \Omega_{a,b,c}$).

The previous order-seven rescaled operators (123), namely $\hat{E}_i$ for $i = 2 \cdots 5$ can actually be seen as special cases of the rescaled (D.2). For instance $\hat{E}_2 = \Omega_{1/2,0,0}$, $\hat{E}_3 = \Omega_{1/3,0,0}$, $\hat{E}_4 = \Omega_{1/4,0,0}$, $\hat{E}_5 = \Omega_{1/6,0,0}$. Note that $\Omega_{0,0,0}$ factors into seven products of order-one operators, namely $\Omega_{0,0,0} = x^7 \cdot (x-1)^2 \cdot \omega_{0,0,0}$, where $\omega_{0,0,0}$ reads:

$$\begin{aligned}\omega_{0,0,0} \;=\;\; & \Bigl(D_x + \frac{8 \cdot x - 6}{(x-1) \cdot x}\Bigr) \Bigl(D_x + \frac{7 \cdot x - 5}{(x-1) \cdot x}\Bigr) \Bigl(D_x + \frac{5 \cdot x - 4}{(x-1) \cdot x}\Bigr) \\ & \times \Bigl(D_x + \frac{4 \cdot x - 3}{(x-1) \cdot x}\Bigr) \Bigl(D_x + \frac{3 \cdot x - 2}{(x-1) \cdot x}\Bigr) \Bigl(D_x + \frac{x - 1}{(x-1) \cdot x}\Bigr) \cdot D_x. \end{aligned} \qquad \text{(D.3)}$$

Appendix E. Yukawa couplings of the operators (146)

Let us recall the operators

$$x \cdot L_{N-1} \;=\; N\,x \cdot (N\,\theta + 1) \cdot (N\,\theta + 2) \,\cdots\, (N\,\theta + N - 1) \;-\; \theta^{N-1}, \qquad \text{(E.1)}$$

annihilating the diagonal of rational functions

$$S_N \;=\; Diag\Bigl(\frac{1}{1 - x_1 - x_2 \,\cdots\, - x_N}\Bigr) \;=\; \sum_{k=0}^{\infty} \frac{(k\,N)!}{(k!)^N} \cdot x^k. \qquad \text{(E.2)}$$

and let us also recall, for these higher orders operators, the "higher order" Yukawa couplings (C.5), (see Appendix C.1 in [18]), one gets the following series expansions.

For L_6 one has two independent Yukawa couplings:

$$\begin{aligned} K_3(L_6) \;=\;\; & 1 + 10097920\,x + 381994497763200\,x^2 + 16633254043776570088000\,x^3 \\ & + 775506882960998615640344320000\,x^4 \\ & + 37663047736228445917647206103076800000\,x^5 + \;\cdots, \\ K_4(L_6) \;=\;\; & 1 + 37273810\,x + 1993144925004100\,x^2 + 110716785445910533561000\,x^3 \\ & + 6240527867851744863088075810000\,x^4 \\ & \quad + 354307497308094243698303790171562900000\,x^5 + \;\cdots \end{aligned}$$

† See operator P_1 in [30].

For L_7 one also has two independent Yukawa couplings:

$$\begin{aligned} K_3(L_7) \;=\;& 1 + 1998080\,x + 17805741956352\,x^2 + 194576429723517255680\,x^3 \\ &+ 2352770839522203863766605056\,x^4 \\ &\quad + 30251355556001122775209879097376768\,x^5 + \;\cdots, \\ K_4(L_7) \;=\;& 1 + 8684032\,x + 117020081027584\,x^2 + 1699286765410547138560\,x^3 \\ &+ 25562078087040978837930064384\,x^4 \\ &\quad + 392649379685173887316823248478666752\,x^5 + \;\cdots, \end{aligned}$$

For L_8 one has three independent Yukawa couplings:

$$\begin{aligned} K_3(L_8) \;=\;& 1 + 28165644\,x + 4505049006911820\,x^2 + 956135990658824836437024\,x^3 \\ &+ 233949266493282926229755622721356\,x^4 \\ &\quad + 62439728262268133355948266259742771574160\,x^5 + \;\cdots, \\ K_4(L_8) \;=\;& 1 + 141215076\,x + 35931446901528372\,x^2 + 10427983259188646965239144\,x^3 \\ &+ 3234974169704568107122039167181620\,x^4 \\ &\quad + 1045613138106144593968102802858498597502432\,x^5 + \;\cdots, \\ K_5(L_8) \;=\;& 1 + 373047525\,x + 145724830276964841\,x^2 + 56876157804695752594934289\,x^3 \\ &+ 22158779200978185414267897869823861\,x^4 \\ &\quad + 8621411505524839637858169288895426578119277\,x^5 + \;\cdots, \end{aligned}$$

For L_9 one also has three independent Yukawa couplings:

$$\begin{aligned} K_3(L_9) \;=\;& 1 + 412077600\,x + 1289641316659740000\,x^2 + 5866947627331695510672000000\,x^3 \\ &+ 32243888417489985271109666517337500000\,x^4 \\ &\quad + 198873128042921363581235281228819150221047577600\,x^5 + \;\cdots, \\ K_4(L_9) \;=\;& 1 + 2351650400\,x + 12183365592555300000\,x^2 + 77390125534275385218992000000\,x^3 \\ &+ 546304879451395256841297089282462500000\,x^4 \\ &\quad + 4116981156738304057315870224531007355517746150400\,x^5 + \;\cdots, \\ K_5(L_9) \;=\;& 1 + 7068028000\,x + 59007489670532260000\,x^2 \\ &+ 517997397361455559649200000000\,x^3 \\ &\quad + 4673650695052899977682502065243662500000\,x^4 \\ &\qquad + 42920054481728604570276319721869711743549574528000\,x^5 + \;\cdots, \end{aligned}$$

For L_{10} one has four independent Yukawa couplings:

$$\begin{aligned} K_3(L_{10}) \;=\;& 1 + 6309779256\,x + 420625737971786884680\,x^2 \\ &+ 45079261240559432713629269385600\,x^3 \\ &\quad + 6138317582117485921071955303191615926205000\,x^4 \\ &\qquad + 96658525320033652731187168658547186232293132446266425 6\,x^5 + \;\cdots, \end{aligned}$$

$$
\begin{aligned}
K_4(L_{10}) \; = \; & 1 + 40564413164\, x + 4622591813564354886036\, x^2 \\
& + 702661675261046914875321464472560\, x^3 \\
& + 124030582023288017842696295577576233194974100\, x^4 \\
& + 240070596480714958890258077251001001284898992609221316 64\, x^5 + \cdots, \\
K_5(L_{10}) \; = \; & 1 + 138021017970\, x + 26412005921656543623886\, x^2 \\
& + 5668431522859800858070100130586968\, x^3 \\
& + 1295519605238134595212195097688057025652309166\, x^4 \\
& + 308199874468915161097074199781704462799392654807438814820\, x^5 + \cdots, \\
K_6(L_{10}) \; = \; & 1 + 317814173215\, x + 94769164483661457482561\, x^2 \\
& + 27681891546591722088579495934023171\, x^3 \\
& + 8013406679969563547817150713512983722786791981\, x^4 \\
& + 230888267014233670780783138418632939667720657070074127 5495\, x^5 + \cdots
\end{aligned}
$$

Remark: Note that, we have relations between these Yukawa couplings and the Yukawa couplings of exterior or symmetric powers of the same operators. For instance the exterior square of L_4 is an irreducible order five operator of nome

$$
\begin{aligned}
q \; = \; & x + 1345\, x^2 + 2552775\, x^3 + 5602757375\, x^4 + 13320846541250\, x^5 \\
& + 33314508430778394\, x^6 + 86273174430421418330\, x^7 \\
& + 229182120170130009397850\, x^8 \\
& + 620754459813846824189800125\, x^9 + \cdots
\end{aligned}
$$

and of Yukawa couplings:

$$
\begin{aligned}
K_3(Ext^2(L_4)) \; = \; & 1 - 575\, x - 1087500\, x^2 - 2357466250\, x^3 - 5515348543750\, x^4 \\
& - 13549400590159950\, x^5 - 34443162481829737500\, x^6 \\
& - 89801360565832417275000\, x^7 - 238760519646901921788093750\, x^8 \\
& - 644794600714076957552558593750\, x^9 + \cdots \\
K_4(Ext^2(L_4)) \; = \; & 1 - 1725\, x - 2270625\, x^2 - 3510633125\, x^3 - 5943482381250\, x^4 \\
& - 10616175881261100\, x^5 - 19525058497227228750\, x^6 \\
& - 36235211885062595043750\, x^7 - 66371326425035067092906250\, x^8 \\
& - 116122894233894656970457656250\, x^9 + \cdots \\
K_5(Ext^2(L_4)) \; = \; & 1 - 2875\, x - 2131250\, x^2 - 1182175000\, x^3 + 1120605484375\, x^4 \\
& + 7242481554278375\, x^5 + 23221270993221987500\, x^6 \\
& + 64082729395471190031250\, x^7 \\
& + 167312972891732043150312500\, x^8 \\
& + 426590930940677272029245312500\, x^9 + \cdots
\end{aligned}
$$

One has the relations

$$
K_4(Ext^2(L_4)) \; = \; K_3(Ext^2(L_4))^3, \qquad K_5(Ext^2(L_4)) \; = \; K_3(Ext^2(L_4))^5,
$$

and one notes the following relation between the Yukawa coupling of L_4 with the Yukawa coupling of its exterior square:

$$
K_3(L_4) \; = \; \frac{1}{K_3(Ext^2(L_4))^2}. \tag{E.3}
$$

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